MAT120: Lecture 16 Handout

Mathematics and Finance

Last lecture we studied hypothesis testing, which used normal distributions, sampling distributions, and probability to make decisions from data. In this lecture we begin a new block of the course by asking how mathematics appears inside money, finance, uncertainty, and risk. The goal is not to turn you into a banker, but to show that the modelling ideas, probability tools, and statistical concepts from earlier in the course all reappear in ordinary financial life. We will move from barter and the meaning of money, to percentages and interest, then to expected value, risk, volatility, and diversification.

1 What is Money?

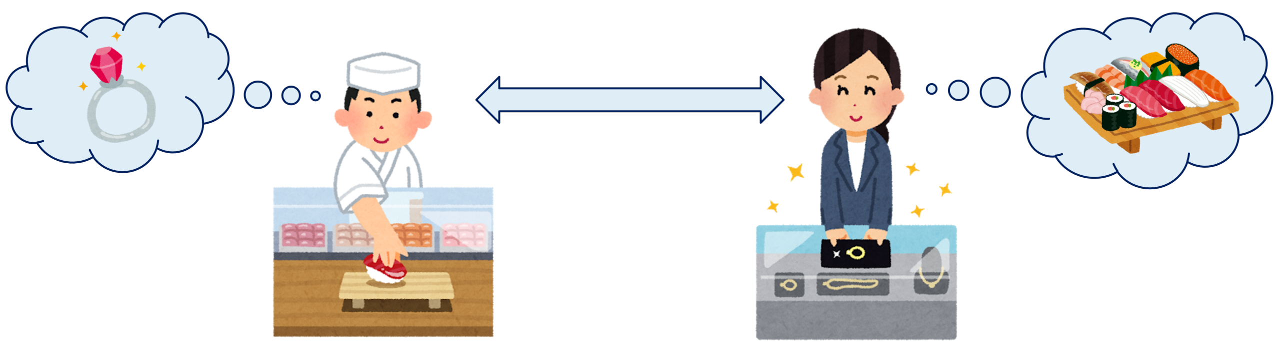
Before talking about finance, it is worth first asking what money actually *is*. A common way to understand money is to see it as solving problems that arise when bartering. In a barter system, two people directly exchange goods: bread for fish, a tool for clothing, and so on. At first this may seem natural, but the system quickly runs into practical difficulties.

1.1 Three problems with barter

A barter economy suffers from at least three important problems.

Problem 1: Suppose that there are two people looking to barter.

- a sushi chef would like to propose to his partner, and
- a hungry jeweller who happens to make shiny engagement rings.



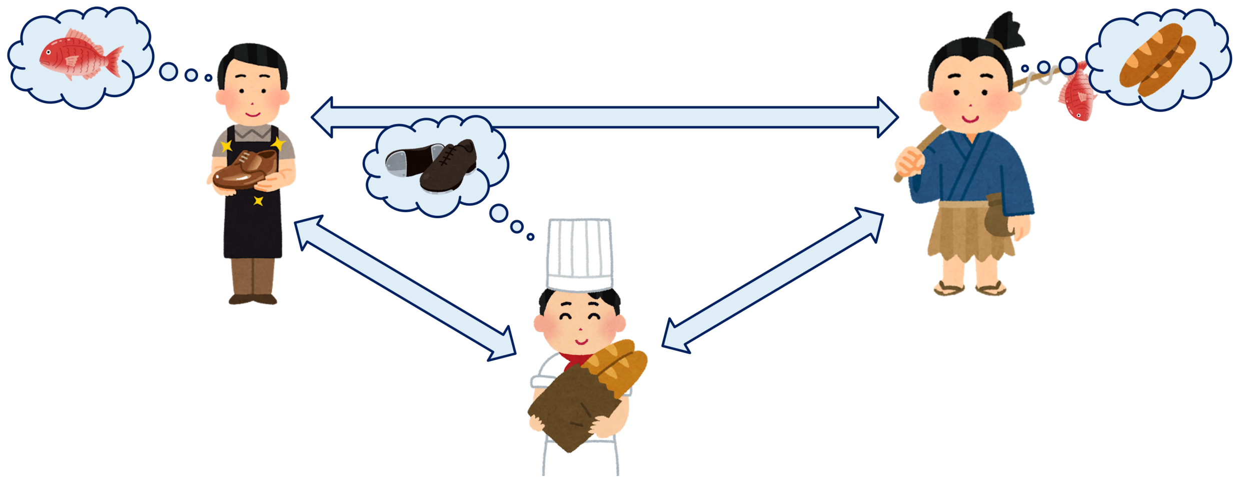
The jeweller enters the sushi restaurant seeking food, and the chef happens to mention that he is looking for a shiny new ring. The chef notices that the jeweller has a ring spare, and also notices that she looks quite hungry. So, he offers her a plate of sushi in exchange for the ring. Will they trade? The answer is *no*, because the ring is clearly much more valuable than a plate of sushi.

A First Problem with Bartering

Two people may value their objects differently. If one person believes an item is worth far more than the other person does, the trade will fail.

Problem 2: suppose now that there are three people looking to trade:

- A fisherman, who is in need of some bread,
- A baker, who is in need of a new pair of shoes, and
- A shoemaker, who is in need of a delicious fish.

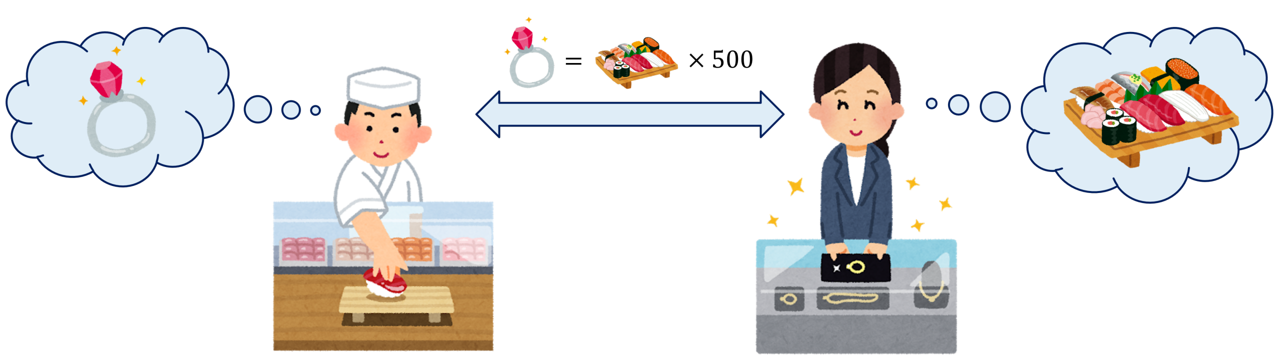


It seems that these three people could easily trade the items, resulting in them all being happy. However, in reality they will probably *not* trade, because each person must want what the next person has at the same time.

A Second Problem with Bartering

Bartering becomes harder to organise when trade requires groups involving more than 2 people. There is no guarantee that a group of people will all have matching wants at the same time.

Problem 3: Let's return to the chef and the jeweller. Suppose that they communicate about the trade, and they both agree that a shiny new ring is worth about 500 plates of sushi:



So, the chef offers 500 plates of sushi in exchange for the ring. In this situation, they *still won't trade*, because although the value of the ring now matches the amount of sushi, the jeweller still realises that in two days' time the sushi will be rotten and therefore worthless. This illustrates a third problem of barter.

A Third Problem with Bartering

The perceived value of an object can change over time. A trade that seems reasonable today may not seem reasonable tomorrow.

1.2 The Function of Money

Amongst other things, money seeks to resolve the three problems mentioned above.

Three core functions of money

Modern money can be understood in three main ways:

- **Medium of exchange:** money allows people to trade indirectly.
- **Unit of account:** money provides a common numerical language for value.
- **Store of value:** money can hold purchasing power over time, at least approximately.

Once we assign prices to objects, barter becomes easier to organise. Instead of asking whether a ring is "equal" to a sushi dinner, we instead ask whether their prices are compatible. In this sense, money is a way of *quantifying value*, even if that value is never perfectly objective.

Older societies used money in more concrete forms: coins, precious metals, shells, grain, and other recognised objects. In the modern day, money is much more abstract, and much of our money does not exist as physical cash, but instead as abstract numbers inside bank accounts, databases and digital systems. But, the underlying function of money stays the same: money is used to store and keep track of value.

In this lecture, it is helpful to think of money as a kind of *social energy.* It can be stored, transferred, and converted into other forms. However, unlike physical energy, money is not a purely natural quantity. Modern money can be created or destroyed by institutions, and its meaning depends on social trust, law, and collective behaviour. Money, being a social practice, is not purely mathematical.

2 Money and Growth

Mathematics can be used to describe how money changes over time. We will focus mostly on types of *interest*, but it should be noted that mathematical finance covers much more than what we will see. We start with a brief review of percentages.

2.1 Percentages and one-step changes

Recall that the word "percent" literally means "per hundred". So:

$$25\% = \frac{25}{100} = 0.25.$$

If a price increases by rate r, then we multiply by $1 + r$. If a price decreases by rate r, then we multiply by $1 - r$. Therefore:

$$\text{New value} = (\text{Old value})(1 \pm r).$$

Examples.

- A shirt priced at \$120 with a 50% discount becomes

$$120(1 - 0.5) = 120(0.5) = 60.$$

- A house valued at \$200,000 that rises by 10% becomes

$$200{,}000(1 + 0.10) = 220{,}000.$$

A common mistake is to think that a 20% decrease followed by a 20% increase returns us to the starting point. It does not, because the second percentage is applied to a *different base*.

Example: discount then increase

Suppose a camera costs \$1000 and receives a 20% discount. Then:

$$1000(1 - 0.20) = 1000(0.8) = 800.$$

If the discounted price then rises by 20%, the new value is

$$800(1 + 0.20) = 800(1.2) = 960.$$

So the camera does *not* return to its original value. A 20% fall and a 20% rise do not cancel each other out.

2.2 Simple interest

Interest deals with repeated changes in money rather than a single change. The first model is *simple interest*. Here, interest is always calculated from the *original principal* P.

If the annual interest rate is r and the time is t years, then the total amount is

$$A = P(1 + rt).$$

This is a *linear* model, because the growth happens by adding the same amount each year.

Simple interest formula

$$A = P(1 + rt)$$

where:

- P is the principal (starting money),
- r is the annual interest rate written as a decimal,
- t is the time in years,
- A is the accumulated amount.

Worked example: simple interest calculation

Suppose you invest \$1000 at a simple interest rate of 5% for 5 years. Then

$$A = 1000(1 + 0.05 \cdot 5) = 1000(1.25) = 1250.$$

So after 5 years you would have \$1250.

2.3 Compound interest

Simple interest is not the usual model for real banking or investment. In *compound interest*, interest is added to the balance, and future interest is then calculated on the new balance. This creates *exponential* growth:

$$A = P(1 + r)^t.$$

The difference is conceptually important. Linear growth keeps adding the same quantity. Exponential growth keeps multiplying by the same factor.

Compound interest formula

$$A = P(1 + r)^t$$

This is the basic yearly compounding model.

Worked example: compound interest calculation

Suppose you deposit \$1000 at 10% annual compound interest. After 2 years,

$$A = 1000(1.1)^2 = 1000(1.21) = 1210.$$

After 3 years,

$$A = 1000(1.1)^3 = 1000(1.331) = 1331.$$

Notice that the third year adds more than the first year did, because the balance is now larger.

2.4 Compound debt

Compound growth does not only apply to savings – it also applies to debt. For example, if you owe money and the lender adds interest to what you owe, then the amount owed can grow exponentially.

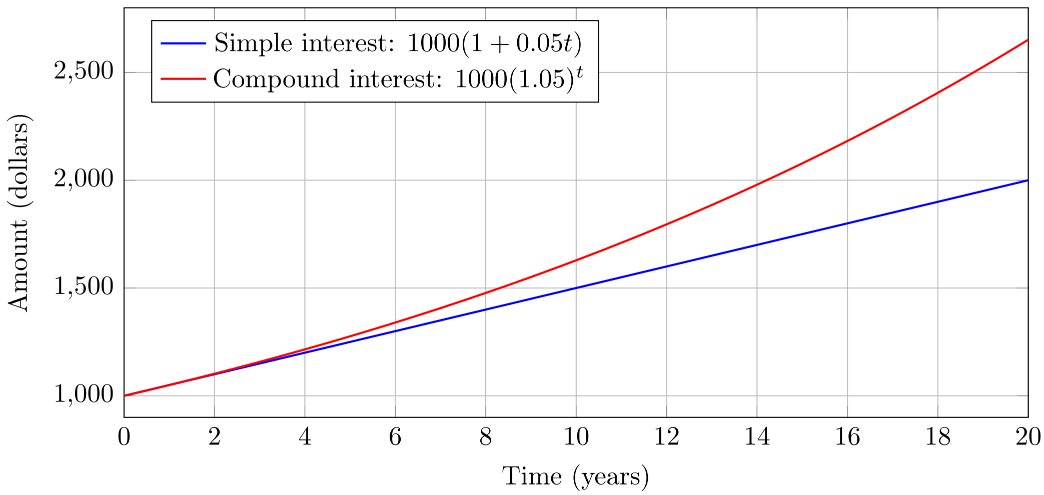
Worked example: compound debt calculation

Suppose that you borrow \$10,000 at 5% annual compound interest and make no repayments for 2 years. Then the amount owed is

$$A = 10{,}000(1.05)^2 = 10{,}000(1.1025) = 11{,}025.$$

So after 2 years you will owe \$11,025 instead of $\$10,000$.

Simple interest and compound interest may start in similar ways, but compound growth eventually pulls away because it repeatedly multiplies. The graph below uses the same starting amount and the same rate of increase, but the long-run behaviour is very different because of the effect of compounding.



2.5 Compounding more than once per year

In principle, interest may be compounded on a more regular basis such as monthly, weekly, or daily. If interest is compounded n times per year, then the formula becomes

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

More frequent compounding usually increases the final amount, because the balance is updated more often.

If compounding becomes more and more frequent, we are led to one of the most important numbers in mathematics, namely $e \approx 2.71828$. The classical limit is

$$\left(1 + \frac{1}{n}\right)^{n} \to e \qquad \text{as } n \to \infty.$$

This is why exponential functions and the number e appear so often in finance, growth, inflation, and many other real-world systems.

Continuous compounding. When interest is compounded continuously, the model becomes

$$A = Pe^{rt}.$$

This is the "smoothest" version of compound growth.

3 Money and Uncertainty

In the previous section we assumed that money was locked into some predictable system of growth. However, in reality money is used in many uncertain systems, such as bets, insurance, financial investments, and so on. Here, the situation is different: it is not *fully understood* what will happen to money over time. Therefore, there is an element of uncertainty, and probabilistic tools are used instead. The central mathematical concept used here is something known as *expected value*.

3.1 Expected value as a weighted average

Expected value is the long-run average outcome of a random situation. We can imagine this as a probabilistic system in which there is money attached to the different outcomes. Expected value can be computed by multiplying each possible outcome by its probability and then adding up the results.

Expected value

If a random variable X can take values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$, then

$$\mathrm{EV}(X) = x_1p_1 + x_2p_2 + \cdots + x_np_n = \sum_{i=1}^{n} x_ip_i.$$

This is just a weighted average – the formula is very similar to the mean of a dataset. It should be noted that expected value is not a prediction of what *must* happen on a single trial. Instead, it is the average result that we would expect to see if the experiment were repeated many times.

3.2 Expected value of a money game

Suppose that you go to the casino and play the following game: you roll a fair die, and

- if the die lands on 5, you win $10,
- otherwise, you lose $1.

Generally, we represent *losing money* by using negative value. So, the two outcomes of this game are 10 with probability 1/6, and −1 with probability 5/6. The expected value of this game can be calculated as:

$$\mathrm{EV} = 10\left(\frac{1}{6}\right) + (-1)\left(\frac{5}{6}\right) = \frac{10}{6} - \frac{5}{6} = \frac{5}{6}.$$

This number is positive, which implies that your average net gain is $\frac{5}{6}$ dollars per play in the long run. So, in the long run, this game would be favourable to the player.

Worked example: casino game expected value

Suppose that you pay \$10 to enter a game. There are three possible net outcomes:

- lose \$10 with probability 0.6,
- break even with probability 0.3,
- win \$40 with probability 0.1.

Then

$$\text{EV} = (-10)(0.6) + (0)(0.3) + (40)(0.1) = -6 + 0 + 4 = -2.$$

So the expected value is \$-2. On average, the player loses \$2 per play.

3.3 Fair and unfair games

A game is called *fair* if its expected value is zero. That means that, in the long run, neither side has an average advantage.

Fairness criterion

$$\text{Fair game} \iff \text{EV}(X) = 0.$$

If $\text{EV}(X) > 0$, the game is favourable to you. If $\text{EV}(X) < 0$, the game is unfavourable to you.

We will now present two examples of this.

Example 1: A coin game

Suppose Alice and Bob flip a fair coin. If the coin lands on heads, Alice gives Bob \$1. If the coin lands on tails, Bob gives Alice \$1.

From Bob's point of view the outcomes are $+1$ with probability $1/2$ and -1 with probability $1/2$, so

$$\text{EV} = 1\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) = 0.$$

So the game is fair.

Example 2: A dice game that is not fair

Now suppose Alice and Bob roll a die. If the die lands on a multiple of 3, Alice gives Bob \$1. If the die does not land on a multiple of 3, Bob gives Alice \$1.

From Bob's perspective, the gain is $+1$ with probability $2/6$ and -1 with probability $4/6$. Therefore

$$\text{EV} = 1\left(\frac{2}{6}\right) + (-1)\left(\frac{4}{6}\right) = -\frac{2}{6} = -\frac{1}{3}.$$

So the game is unfair for Bob.

To make the game fair, suppose Bob instead pays Alice only x dollars when he loses. Then

$$1\left(\frac{2}{6}\right) + (-x)\left(\frac{4}{6}\right) = 0.$$

Solving gives

$$2 - 4x = 0 \qquad \Longrightarrow \qquad x = \frac{1}{2}.$$

So Bob would need to pay only 50 cents when he loses.

3.4 Roulette and the casino

A roulette-style colour game can *feel* fair because red and black seem symmetric. But if the wheel includes an extra green slot, that symmetry breaks.

Suppose a wheel has 37 slots: 18 red, 18 black, and 1 green. You bet \$1 on a colour and win a net gain of \$1 if your colour appears, but lose \$1 otherwise. Then the expected value is

$$\text{EV} = (+1)\left(\frac{18}{37}\right) + (-1)\left(\frac{18}{37}\right) + (-1)\left(\frac{1}{37}\right) = -\frac{1}{37}.$$

So the expected value is negative. The casino has only a small advantage on one play, but over many plays that advantage accumulates.

4 Money and Risk

Expected value is important, but it is not the whole story. Finance is not only about average outcomes. It is also about *risk* and the *management of risk*. Two choices may have the same expected value and yet feel very different because one is much more volatile or much more dangerous.

4.1 Insurance

Insurance is one of the clearest examples showing that people do not make decisions using expected value alone. To illustrate this, consider the following example.

Suppose you own a bicycle worth \$200 and there is a 10% chance that it is stolen during the year. Without insurance, your random loss is:

- \$200 with probability 0.1,
- \$0 with probability 0.9.

So the expected loss is

$$\text{EV(loss)} = 200(0.1) + 0(0.9) = 20.$$

Now suppose an insurance company offers full coverage for \$30.[1] From the purely mathematical perspective of expected value, paying \$30 to avoid an average loss of \$20 is not a good choice. However, many people still buy insurance. Why? Because insurance replaces a risky situation with a more *predictable* one.

[1]Here the term "full coverage" means that the insurance company will pay the entire value of the bicycle, assuming it is stolen.

The concept of insurance

A choice can have negative expected value and still be rational, if it significantly reduces exposure to large losses.

4.2 Volatility

Volatility measures how much a quantity moves up and down. In finance, low volatility means returns stay relatively close to their average, while high volatility means they swing widely. We have seen this idea of "spreading widely" before: this was precisely the notion of *variation* of a dataset. In fact, standard deviation can be used to measure the volatility of investments.

To see this in action, consider the two investment strategies below:

Observation	Strategy A	Strategy B
1	+$100	+$200
2	+$110	-$50
3	+$90	+$100
4	+$100	+$150

Both strategies have the same mean return:

$$\bar{x}_A = \bar{x}_B = 100.$$

However, as you can see, Strategy A is much more reliable and stable, whereas Strategy B is much more volatile. We can quantify this volatility by computing the standard deviation of both.

For Strategy A, the deviations from the mean are $0, 10, -10, 0$, so the sample variance is

$$s_A^2 = \frac{0^2 + 10^2 + (-10)^2 + 0^2}{4 - 1} = \frac{200}{3} \approx 66.67.$$

Taking the square root of this value gives the standard deviation:

$$s_A \approx 8.16.$$

For Strategy B, the deviations are $100, -150, 0, 50$, which are much larger than A's. So the sample variance will also be larger:

$$s_B^2 = \frac{100^2 + (-150)^2 + 0^2 + 50^2}{4 - 1} = \frac{35{,}000}{3} \approx 11{,}666.67.$$

Taking a square root gives B's standard deviation:

$$s_B \approx 108.01.$$

As you can see, although the average returns of both strategies are the same, Strategy B is vastly more volatile.

Interpretation

If you only asked “Which option has the bigger average?”, then A and B would appear equally good. But once you include volatility, Strategy A is much more predictable, whereas Strategy B involves much larger swings. This is why finance is about more than average return. It is also about what level of uncertainty you can tolerate.

4.3 Diversification

Diversification means spreading money across different assets instead of concentrating it all in one place. The idea is simple: if one investment performs badly, another may perform better, so the overall risk can be reduced.

Diversification

“Do not put all your eggs in one basket” is a financial principle as much as a proverb.

Diversification does *not* guarantee profit. Instead, it is a strategy for managing risk by reducing dependence on any single outcome.

Common Investment Strategies

Common examples of investment strategies include:

- government bonds,
- index funds, and
- premium bonds.

These strategies differ in their predictability, expected value and diversification.

A less volatile investment is often more predictable. A more volatile investment may offer bigger gains, but it may also expose you to bigger losses. So instead of asking “which option has the biggest average?”, it is perhaps better to ask:

> Which option can I realistically live with?

5 Formula Summary and Exercises

Formula summary

$$
\begin{aligned}
\text{Percentage change:} \quad & \text{new value} = (\text{old value})(1 \pm r), \\
\text{Simple interest:} \quad & A = P(1 + rt), \\
\text{Compound interest:} \quad & A = P(1 + r)^t, \\
\text{Compounding } n \text{ times/year:} \quad & A = P\left(1 + \frac{r}{n}\right)^{nt}, \\
\text{Continuous compounding:} \quad & A = Pe^{rt}, \\
\text{Expected value:} \quad & \text{EV}(X) = \sum_{i=1}^{n} x_i p_i, \\
\text{Sample variance:} \quad & s^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1}.
\end{aligned}
$$

Exercises

(a) A laptop costs \$1800. It is discounted by 15%, and the discounted price is then increased by 15%. What is the final price? Does it return to \$1800?

(b) You invest \$2500 at 4% simple interest for 6 years. How much money will you have at the end?

(c) You deposit \$3000 at 6% compound interest for 4 years. How much money will you have if interest is compounded yearly?

(d) A game has three possible net outcomes: lose \$5 with probability 0.5, lose \$1 with probability 0.3, and win \$20 with probability 0.2. Find the expected value.

(e) A fair coin is tossed. If heads appears, you win \$3. If tails appears, you lose \$2. Is the game fair, favourable, or unfavourable?

(f) Consider the data sets

$$A = (8, 10, 12, 10), \qquad B = (0, 10, 20, 10).$$

Show that they have the same mean, but different volatility.

Selected answers

(a)

$$1800(0.85)(1.15) = 1759.50.$$

So the final price is \$1759.50, not \$1800.

(b)

$$A = 2500(1 + 0.04 \cdot 6) = 2500(1.24) = 3100.$$

(c)

$$A = 3000(1.06)^4 \approx 3000(1.26248) \approx 3787.43.$$

(d)

$$\text{EV} = (-5)(0.5) + (-1)(0.3) + (20)(0.2) = -2.5 - 0.3 + 4 = 1.2.$$

So the expected value is \$1.20.

(e)

$$\text{EV} = 3\left(\frac{1}{2}\right) + (-2)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

So the game is favourable to the player.

(f) Both means are 10, and the sample variances are $s_A^2 = \frac{8}{3}$ and $s_B^2 = \frac{200}{3}$. So B has larger variance and is therefore more volatile.