# MAT120: Lecture 18 Handout
*Mathematics and Games*

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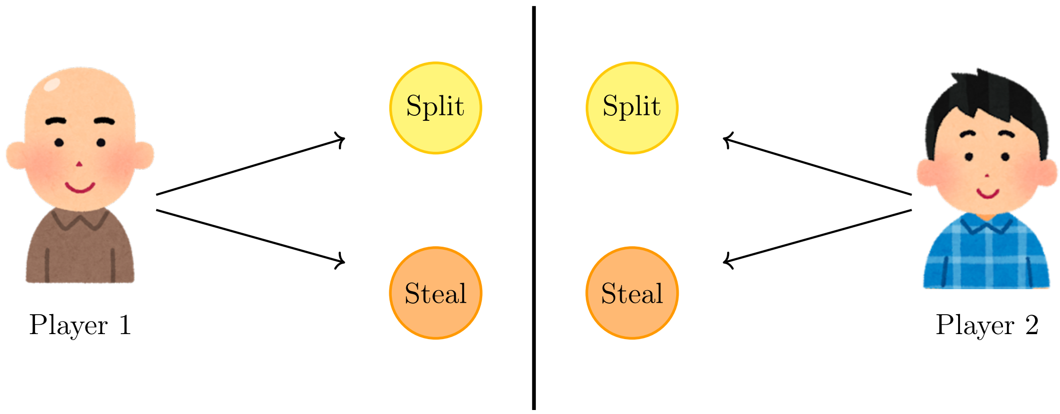
Last lecture we studied mathematics and social choice, where the main question was how a group should combine many individual preferences into one collective decision. In this lecture we stay inside the "mathematics in society" block, but we now shift from voting to *strategy*. The central question of game theory is this: what is the best way for me to act, given the rules of the situation, and given that other people are also thinking strategically? We will begin with a game show example, then introduce the basic language of players, strategies, outcomes, and payoffs. After that we will study several famous examples, including the Prisoner's Dilemma, the game of averages, Battle of the Sexes, and Hotelling's Law. Finally, we will define *Nash equilibria*, which are the main stability concepts of elementary game theory.

## 1 Introduction

Golden Balls was a British TV show that ran from 2007 to 2009. In the show, a group of participants had to compete in a variety of challenges which caused members of the group to be successively eliminated. Along the way, the competitors built up a cash prize, typically around the equivalent of tens of thousands of dollars. In the final round of the competition, almost all of the participants had been eliminated and only two remained. They are presented with a final game: the two contestants have two *golden balls* in front of them. Inside these balls are the words "split" and "steal". The players can look inside their pair of golden balls, but they cannot see inside the balls of the other player:




These two players then have to compete for the entire prize money. They are presented with the following rules:

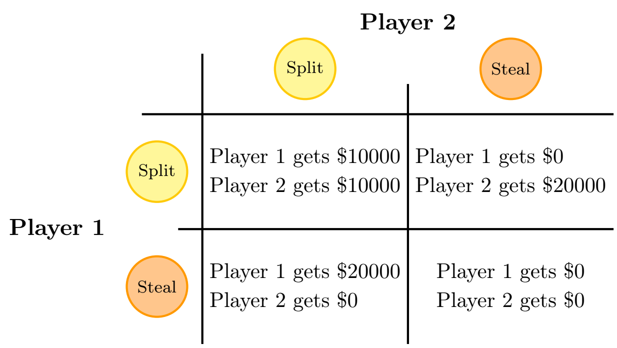
1. if both players choose **split**, then they divide the money equally;
2. if Player 1 chooses **split** and Player 2 chooses **steal**, then Player 2 takes everything;
3. if Player 1 chooses **steal** and Player 2 chooses **split**, then Player 1 takes everything;
4. if both players choose **steal**, then they both get nothing.

Interestingly, the players are then allowed to talk to each other and try to create a strategy. Typically, one of two things would happen:

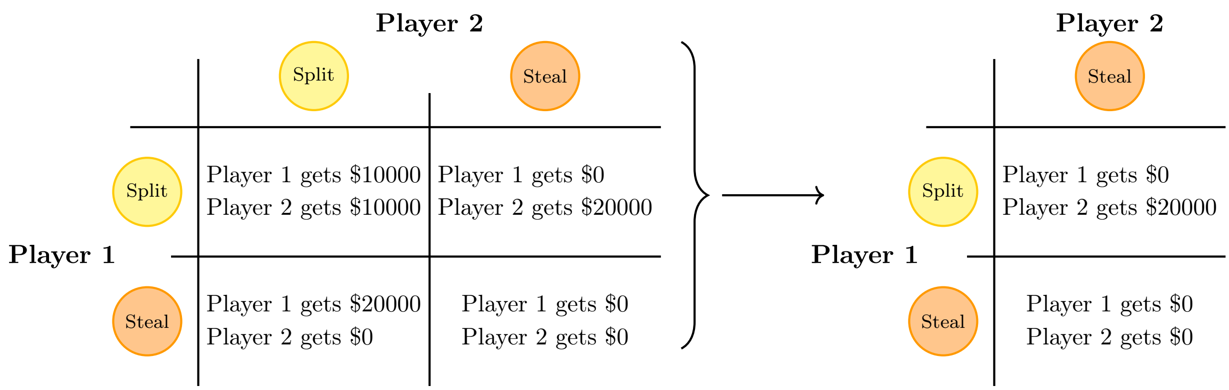
- the two players would come to a mutual agreement to both pick the "split" ball, since it is in their best mutual interest to do so,
- the two players would come to a mutual agreement to both pick "split", but then one or both of the players would betray the agreement and actually pick the "steal" ball.

However, in one episode, there was a competitor who devised a remarkable strategy: Player 2 insisted to Player 1 that he would be picking the *steal* ball, no matter what. He then claimed that he would split the actual prize money with the other competitor after the show. After some back-and-forth arguing, Player 1 picked the split ball, and amazingly so did Player 2.

To see the brilliance of this strategy, let's try to break down the possibilities. For the sake of convenience, let's assume that the prize pot was equal to about \$20000. According to the rules, there are four possible outcomes to this game:




In this situation, Player 2 precisely understood the rules of the game. He knew that he could not trust Player 1 to reliably pick the "split" ball and therefore there was a chance that Player 2 would leave the show with nothing. To minimize this risk of betrayal, Player 2 decided to insist on picking the steal ball, which effectively removed one half of the possible outcomes:




By claiming to Player 1 that he would split the money after the show, Player 2 is effectively saying to Player 1:

> "You have a 0% chance of receiving any money if you pick 'steal'. However, if you pick 'split' then you have a non-zero chance of receiving half the money after the show."

By opting for this strategy, Player 2 was trying to minimize the chance that Player 1 picks the "steal" ball. This worked, and Player 2 therefore decided to split in the end, now that he was confident that the money would not be stolen.

At first glance, this looks like a problem about trust, morality, or human psychology. In one sense, it is all of those things. But in another sense it is something much more precise: a mathematical problem about how rational players should behave when the outcome depends on *both* players' decisions. That is exactly the starting point of game theory.

**Game theory**

Game theory is the mathematical study of **strategy**. It asks how rational decision-makers should act when their outcomes depend not only on their own choices, but also on the choices of other decision-makers.

A first attempt at the main question of game theory would be:

> What is the best way for me to act, given the rules of the game?

However, this is incomplete. In reality, strategies should also be informed by what the other players are planning to do. So the fuller question is:

> What is the best way for me to act, given the rules of the game, and given that the other players are also forming their own strategies?

## 2 The Language of Games

In game theory, a *game* is not restricted to board games or video games. Instead, it is any situation involving several decision-makers whose outcomes depend on one another. Examples can be found in economics, political negotiation, biological competition, pricing wars, voting, and military planning.

**What is a game?**

A game is a mathematical model of strategic interaction. It tells us:

- who the players are,
- what choices each player can make,
- and what each player receives for every combination of choices.

### 2.1 Key terminology

The following four words form the core vocabulary of introductory game theory.

**Key terminology**

- **Player:** a person or group making a decision.
- **Strategy:** a choice, or more generally a plan of action.
- **Outcome:** what happens after all players have chosen.
- **Payoff:** a numerical description of how good the outcome is for each player.

The word "payoff" should not be interpreted too narrowly. In some cases, it can literally mean money, but often it only means "a number that describes how desirable the outcome is". In some models the payoff may represent profit, in others it may represent happiness, years in prison, political advantage, fitness, or any other quantity that measures preference.

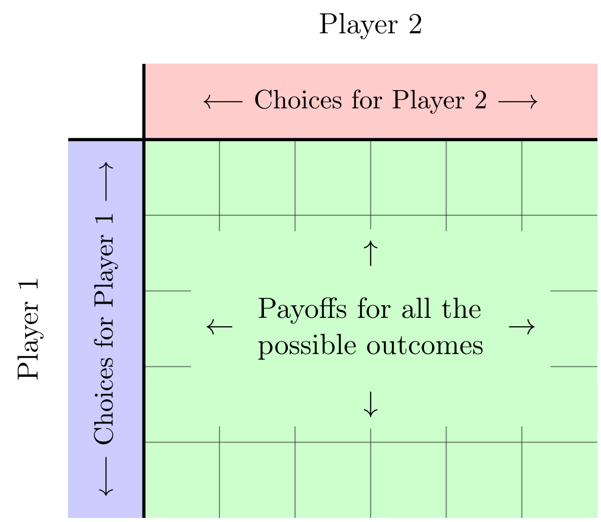
### 2.2 Visualising a game

At a basic level, a game can be viewed as a process with three stages:

The strategic difficulty appears in the middle stage. Each player must choose a strategy while anticipating what the others may or may not do.

## 2.3 Payoff matrices

For small games with two players, it is convenient to organise the possible outcomes into a *payoff matrix*. This is a grid of information which details the payoff associated with every possible outcome of the game. In a sense, it contains information about all the things that can possibly happen within the game. Schematically, it is written as follows:




Here, the rows correspond to Player 1's choices and the columns correspond to Player 2's choices. The green central area consists of a grid that records the payoffs resulting from each pair of choices.

**Reading a payoff matrix.** If a box contains $(a, b)$, then $a$ is Player 1's payoff and $b$ is Player 2's payoff. The first number always belongs to the row player, and the second number always belongs to the column player.

In fact, we saw a prototype of a payoff matrix in the introduction. In less pretty notation, the payoff matrix associated with the split/steal game can be written as follows:

| | | **Player 2** | |
|---|---|---|---|
| | | **Split** | **Steal** |
| **Player 1** | **Split** | $(10000, 10000)$ | $(0, 20000)$ |
| | **Steal** | $(20000, 0)$ | $(0, 0)$ |

## 2.4 Dominant strategies

Sometimes a player has one strategy that is best no matter what the other players do. Such a strategy is called a *dominant strategy*. It is the strategy that dominates all the alternatives.

**Dominant strategy**

A strategy is called **dominant** for a player if it gives that player a payoff at least as good as every other available strategy, for every possible choice of the other players. If it is always strictly better, we call it a **strictly dominant** strategy.

In the formal *Golden Balls* payoff matrix above, **steal** is a dominant strategy for each player, because:

- if the other player chooses split, then stealing gives a larger payoff than splitting, and
- if the other player chooses steal, then stealing gives the same payoff as splitting.

So, from the narrow point of view of the formal matrix, the "rational" move seems to be steal. This does *not* mean real human beings will always do it. It only means that, inside the mathematical model, steal dominates split.

# 3 Examples of Games

We now review some classical examples of games. These are:

1. The Prisoner's Dilemma,
2. The Game of Averages,
3. The Battle of the Sexes, and
4. Hotelling's Law.

As we will see, each game illustrates a different kind of strategic problem.

## 3.1 Example 1: The Prisoner's Dilemma

Suppose that you and your friend are arrested. The police know that a crime was committed, but they do not know which person played the main role. So, you are separated from your friend and you are given the following deal:

- if you snitch while the other person stays silent, you go free and the other person gets 10 years in prison;
- if you stay silent while the other person snitches, you get 10 years in prison and the other person goes free;
- if both of you stay silent, you both get 2 years in prison;
- if both of you snitch, you both get 5 years in prison.

The two suspects are not allowed to communicate beforehand. The matrix for this game is below:

| | | **Your friend** | |
|---|---|---|---|
| | | **Don't snitch** | **Snitch** |
| **You** | **Don't snitch** | $(2,2)$ | $(10,0)$ |
| | **Snitch** | $(0,10)$ | $(5,5)$ |

Here the entries record years in prison, so lower numbers are better.

Let's consider the two choices available to you:

1. If your friend decides not to snitch, then you are constrained to the first column of the payoff matrix. Here, you have a choice between receiving 2 years in prison for not snitching, or 0 years in prison for snitching. Of these two options, it makes sense to snitch.
2. If your friend decides to snitch, then you are constrained to the second column of the payoff matrix. Here, you now have a choice between not snitching and therefore receiving 10 years in prison, or snitching and receiving only 5. From your perspective, it makes sense to snitch in this scenario.

In both possible situations, deciding to snitch is actually better for you. In other words, snitching is a dominant strategy.

**Conclusion for the Prisoner's Dilemma**

**Snitching** is a dominant strategy for both players. So the natural strategic prediction is that both players will snitch, even though both would be better off if they could somehow coordinate on staying silent.

This is the "dilemma" in the Prisoner's Dilemma. According to individual incentives, both players ought to snitch on each other, despite there being a better alternative.

### 3.2 Example 2: A game of averages

Suppose that every student in a class chooses a real number between 0 and 100. Then the class average is calculated, and the winner is the student whose number is closest to $\frac{2}{3}$ of that average. What is the best number to pick here?

At first glance, it seems as though many numbers might be reasonable for our guess. However, consider the following line of reasoning:

- If everybody picked 100, then $\frac{2}{3}$ of the average would only be about 66.7. So choosing 100 would clearly be too large. In fact, anything above around 66.6 looks bad, since there is no better chance of being close to the real average.
- If we have realized this, we understand that everybody else in the class may also figure this out. Therefore, we can conclude that nobody will pick a number above 66.6.
- So, we have reduced our problem to the problem of everybody picking a number between 0 and 66.6. Again, we notice that if everybody were to pick the maximum value of 66.6 then the average would be 66.6 itself and therefore the desired number would be 44.4. We conclude that it doesn't make sense to pick a number more than 44.4.
- Again, we reason that everybody else playing the game could have also realized this, meaning that nobody will pick a number greater than 44.4.

We can repeat this line of reasoning over and over, and keep bringing the ideal guess lower and lower until we conclude that the best possible guess is actually to select the number 0.

**Iterated reasoning**

The game of averages is a good example of **iterated strategic reasoning**. You do not only ask "what should I do?" You ask "what should I do, knowing that the other players are asking exactly the same question?"

### 3.3 Example 3: Battle of the Sexes

A classical coordination game is the *Battle of the Sexes*. Imagine a married couple deciding what to do for date night: the wife prefers to go to a ballet recital whereas the husband prefers to go to a boxing match. However, both would still rather go to the same place together than end up apart. A typical payoff matrix for this setup is:

| | | **Husband** | |
|---|---|---|---|
| | | **Ballet** | **Boxing** |
| **Wife** | **Ballet** | $(2,1)$ | $(0,0)$ |
| | **Boxing** | $(0,0)$ | $(1,2)$ |

Note that the exact numbers in this payoff matrix are not particularly important, as long as the preferences are ranked as:

- the wife prefers Ballet together over Boxing together over apart, and
- the husband prefers Boxing together over Ballet together over apart.

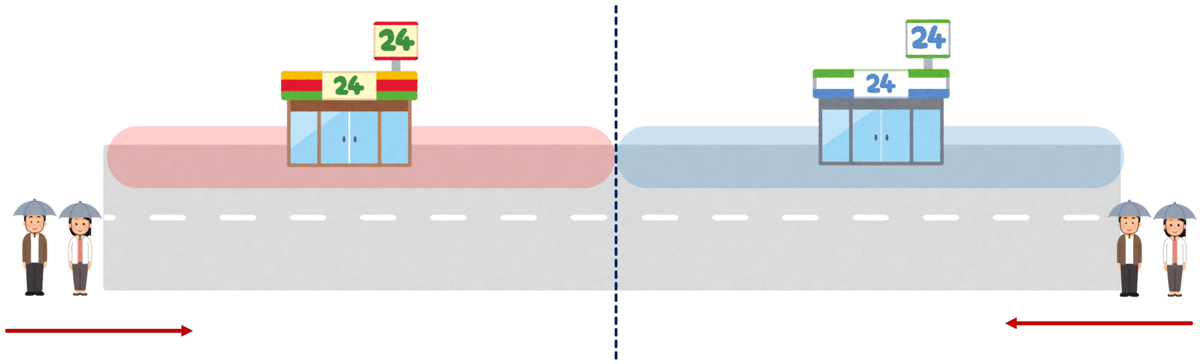
Unlike the Prisoner's Dilemma, neither player has a dominant strategy. For example, from the wife's perspective:

- if the husband is expected to choose Ballet, then Ballet is the wife's best response, and
- if the husband is expected to choose Boxing, then Boxing is the wife's best response.

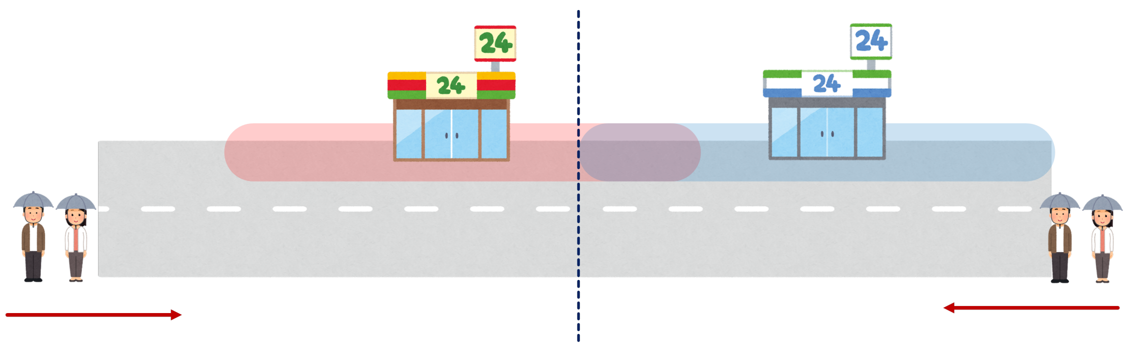
Of course, the same is true for the husband. As we can see here, the main difficulty is *coordination*, not strategic dominance.

### 3.4 Example 4: Hotelling's Law

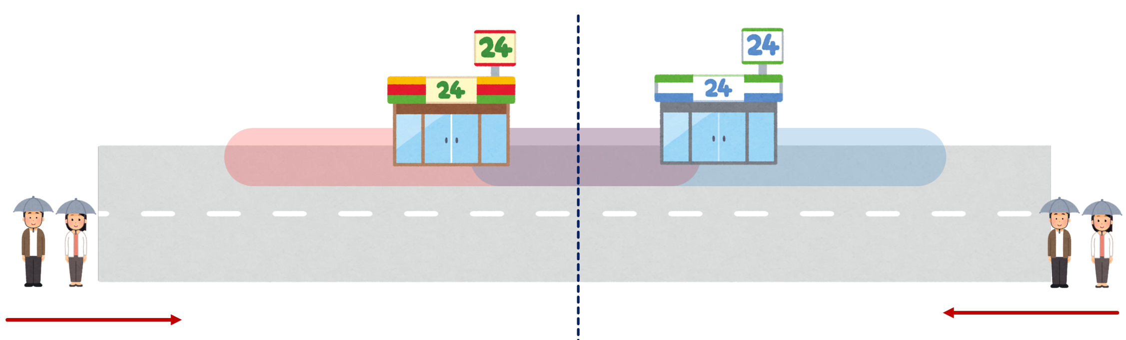
Suppose two convenience stores are deciding where to locate on the same long street. Suppose that the street only has two entrances: from the left, and from the right. At first it might seem ideal for public convenience if one store goes near the left side and the other goes near the right side. That way, customers from the left would go into the first convenience store, and customers from the right would go into the second convenience store:



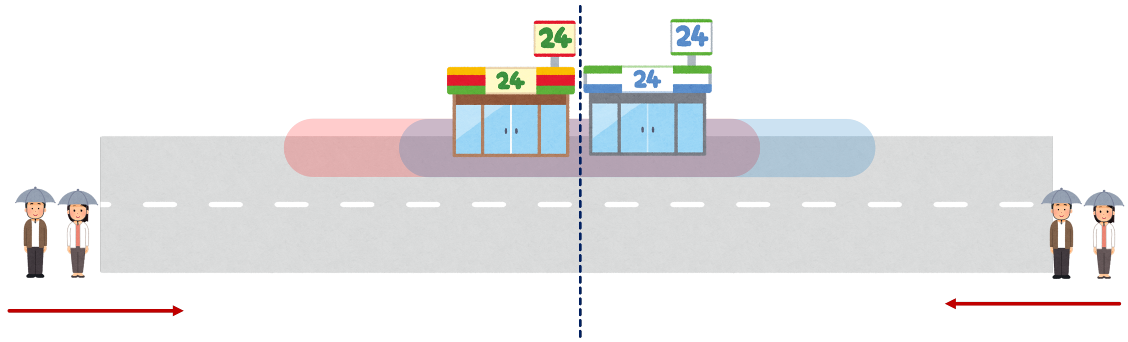

Each store is motivated to capture as many customers as possible. The store on the left realizes that if it were to move closer to the other store, then there would be a chance that it could attract some of the people entering the street from the right, and therefore steal some of Store 2's customer base. Moreover, this can be done without risking the loss of any customers entering from the left, since they will still find Store 1 first.




Of course, Store 2 may also reach the same conclusion. Either by acting first or in retaliation, Store 2 may also move closer to Store 1 and potentially collect some of Store 1's customer base:




Eventually, by repeating this reasoning, both stores will inevitably converge to the centre of the street. This is the only "stable" position on the street, in that neither store can move to another location without giving up some ground to the other store:




In the end, both stores guarantee roughly 50% coverage of the street, but only by clustering together near the middle. So the equilibrium is stable for the stores, but at the cost of public inconvenience.

# 4 Nash Equilibria

A central idea in elementary game theory is that of *equilibrium*. In ordinary language, an equilibrium is a stable point of a system. In game theory, the standard equilibrium concept is named after John Nash, a famous game theorist who introduced this equilibrium concept.

**Nash equilibrium**

A **Nash equilibrium** is a situation in which:

- every player has chosen a strategy,
- each player's strategy is a best response to the other players' strategies, and
- no player can improve their outcome by changing strategy *alone*.

In simple terms, everybody is already doing the best they can, given what everyone else is doing.

## 4.1 Best responses

To understand Nash equilibria properly, we first need the idea of a *best response*. Suppose the other player's choice is temporarily fixed. Among your available strategies, some will give you a larger payoff than others. A best response is simply one of the strategies that maximises your own payoff under that fixed assumption.

So the usual procedure for finding Nash equilibria in a payoff matrix is:

1. fix Player 2's choice and find Player 1's best response in that column;
2. repeat for every column;
3. then fix Player 1's choice and find Player 2's best response in that row;
4. repeat for every row;
5. any box where both players are simultaneously playing best responses is a Nash equilibrium.

### 4.2 A $3 \times 3$ worked example

Consider the following payoff matrix. We will mark Player 1's best responses with a star $*$ and Player 2's best responses with a dagger †.

| | **Choice 1** | **Choice 2** | **Choice 3** |
|---|---|---|---|
| **Choice 1** | $3, 7^{\dagger}$ | $3, 2$ | $1, 1$ |
| **Choice 2** | $2, 4$ | $2, 2$ | $^{*}4, 7^{\dagger}$ |
| **Choice 3** | $^{*}6, 2$ | $^{*}5, 4^{\dagger}$ | $3, 3$ |

Here is the step-by-step procedure:

**Step 1:** mark Player 1's best responses with a $*$ by going through each column of the matrix.

- In Column 1, the first coordinates are $3, 2$ and 6, so the best response is Choice 3.
- In Column 2, the first coordinates are $3, 2$ and 5, so the best response is again Choice 3.
- In Column 3, the first coordinates are $1, 4$ and 3, so the best response is Choice 2.

**Step 2:** mark Player 2's best responses by row.

- In Row 1, the second coordinates are $7, 2, 1$, so the best response is Choice 1.
- In Row 2, the second coordinates are $4, 2, 7$, so the best response is Choice 3.
- In Row 3, the second coordinates are $2, 4, 3$, so the best response is Choice 2.

**Step 3:** find the intersections. The boxes with both a star and a dagger are:

$$(\text{Choice 2}, \text{Choice 3}) \qquad \text{and} \qquad (\text{Choice 3}, \text{Choice 2}).$$

Therefore, this game has **two** Nash equilibria.

**Important point**

A game may have:

- exactly one Nash equilibrium,
- several Nash equilibria,
- no pure-strategy Nash equilibrium at all.

So the goal is never to assume an equilibrium exists and is unique. The goal is to analyse the matrix carefully.

### 4.3 Nash equilibria in the examples from this lecture

We can now revisit the earlier examples from the correct point of view.

- In **Golden Balls**, the formal payoff matrix has a Nash equilibrium at **(Steal, Steal)**. Since steal is dominant for both players, this equilibrium is immediate.
- In the **Prisoner's Dilemma**, the Nash equilibrium is **(Snitch, Snitch)** for the same reason.
- In the **game of averages**, the equilibrium prediction is that everyone chooses 0.

- In **Battle of the Sexes**, there are **two** pure-strategy Nash equilibria: **(Ballet, Ballet)** and **(Boxing, Boxing)**.
- In **Hotelling's Law**, the equilibrium occurs when both stores locate in the middle region of the street.

The main lesson is that a Nash equilibrium is not necessarily the morally best outcome, the socially best outcome, or even the most pleasant outcome. It is simply a *stable* outcome.

## 5 Exercises

**Question 1. Basic vocabulary warm-up**

Fill in the missing word from the list below.

*player, strategy, payoff, outcome, Nash equilibrium, dominant strategy*

**(a)** A number that measures how good an outcome is for a player is called a ____________________.

**(b)** A choice or plan of action in a game is called a ____________________.

**(c)** A situation where nobody can improve their result by changing their choice alone is called a ____________________.

**(d)** A strategy that is best no matter what the other player does is called a ____________________.

**(e)** What happens after everyone has chosen is called the ____________________.

**Solutions to Question 1**

**(a)** **payoff**
**(b)** **strategy**
**(c)** **Nash equilibrium**
**(d)** **dominant strategy**
**(e)** **outcome**

**Question 2. Reading a payoff matrix**

Consider the payoff matrix below.

| | | Player 2 | |
|---|---|---|---|
| | | **Left** | **Right** |
| **Player 1** | **Up** | $(4,3)$ | $(1,5)$ |
| | **Down** | $(6,2)$ | $(2,4)$ |

The first number is Player 1's payoff and the second number is Player 2's payoff.

- **(a)** If Player 1 chooses Up and Player 2 chooses Right, what is the outcome?
- **(b)** If Player 2 chooses Left, which choice gives Player 1 the larger payoff?
- **(c)** If Player 1 chooses Down, which choice gives Player 2 the larger payoff?
- **(d)** Does Player 1 have a dominant strategy? Explain briefly.
- **(e)** Find the Nash equilibrium, if there is one.

**Solutions to Question 2**

- **(a)** The outcome is $(1,5)$. So Player 1 gets 1 and Player 2 gets 5.
- **(b)** If Player 2 chooses Left, then Player 1 compares 4 and 6. So Player 1 should choose **Down**.
- **(c)** If Player 1 chooses Down, then Player 2 compares 2 and 4. So Player 2 should choose **Right**.
- **(d)** Yes. Player 1 has the dominant strategy **Down**, because if Player 2 chooses Left then $6 > 4$, and if Player 2 chooses Right then $2 > 1$.
- **(e)** The Nash equilibrium is **(Down, Right)**, since Down is Player 1's best response to Right and Right is Player 2's best response to Down.

**Question 3. Prisoner's Dilemma reasoning**

Two suspects must each choose either **Silent** or **Snitch**. Their payoffs are shown below.

| | **Friend: Silent** | **Friend: Snitch** |
|---|---|---|
| **You: Silent** | $(2,2)$ | $(0,10)$ |
| **You: Snitch** | $(10,0)$ | $(5,5)$ |

- **(a)** If your friend stays Silent, what should you choose?
- **(b)** If your friend chooses Snitch, what should you choose?
- **(c)** What is your dominant strategy?
- **(d)** By symmetry, what is your friend's dominant strategy?
- **(e)** What is the Nash equilibrium?

**Solutions to Question 3**

- **(a)** If your friend stays Silent, you should choose **Snitch**, because $10 > 2$.
- **(b)** If your friend chooses Snitch, you should again choose **Snitch**, because $5 > 0$.
- **(c)** Your dominant strategy is **Snitch**.
- **(d)** By symmetry, your friend's dominant strategy is also **Snitch**.
- **(e)** The Nash equilibrium is **(Snitch, Snitch)**.