MAT120 Lecture 4 Handout
Arithmetic

Contents

Last lecture we were working with formal logic and truth tables. In this lecture we begin the algebra block of the course by returning to arithmetic. Although arithmetic may feel very basic, these techniques are the foundation for much of what comes later: counting, equations, modelling, finance, probability and statistics all rely on the same basic numerical language. Our goal today is not to make arithmetic look difficult, but to organise familiar ideas clearly. We will review the number line, inequalities, arithmetic operations, fractions, decimals and percentages.

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1 The Number Line

We will start with an almost philosophical question: what is a number? The mathematicians of old have provided several answers to this question, and we will not review them here. Instead, we will take the following approach: a number is anything that we need it to be.

Let's start small, with the idea of *quantity*. At some point in human history, we noticed that we can have more or less of objects of the same type. For example, having three coconuts is more than having two coconuts, and having two bananas is less than having four bananas. At some point we made a great leap forward in our cognition: we noticed that two coconuts and two bananas have something in common and we abstracted away from physical objects into the realm of mathematics and introduced the concept of a number. Numbers like 1,2,3 and so on are abstract representations of quantity. Later on, we eventually realised that we can also have *nothing* of something, and therefore we also introduced the number 0 to represent that. Mathematicians refer to these "counting numbers" $0, 1, 2, 3, \cdots$ as *natural numbers*, and we use the funny letter $\mathbb{N}$ to denote the collection of all of them.

In an obvious sense, having 2 of something is more than having 1 of something, so there is a natural way to order all of these natural numbers:

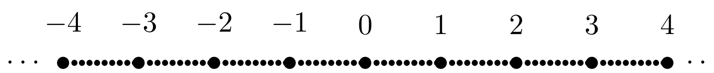
$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ \bullet & \bullet & \bullet & \bullet & \bullet \cdots \end{array}$$

Moreover, we noticed that these quantity numbers have an operation of addition: if I have 3 bananas and I collect 2 more, then I have 5 bananas in total. After a while, we realised that alongside adding quantities, we would also like an opposite operation of *removing* quantities. This is, of course, subtraction. One way to understand subtraction is by way of *negative numbers*: if we have 3 bananas and we remove one from our collection, then it's like adding -1 to our collection. We can keep doing this of course, and therefore we discover that there is another copy of our natural numbers, this time pointing in the opposite direction:

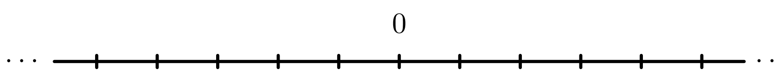
$$\begin{array}{ccccccccccc} & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \\ \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \end{array}$$

We call this collection the *integers*, and we use the letter $\mathbb{Z}$ to denote the collection of all of them.

After a while, we can notice that individual objects in the real world can also be *divided*, for instance if we have a birthday cake, we may need to cut it up and give out individual parts of it to separate people. With this in mind, we need a way to represent *ratios* of quantities too, such as half of something. This motivates the general definition of a *rational number*, that is, a number that can be written as a ratio of two integers: $\frac{a}{b}$, where $b \neq 0$. Mathematicians write the collection of all rational numbers using the letter $\mathbb{Q}$. Since there are many integers, there are also many many ways to write combinations of them into a ratio. So, if we were to write all of these onto our drawing of all the numbers, things would start to look something like this:



As a matter of fact, these rational numbers are densely packed around the integers: no matter how far you zoom in on an integer, you will always find a rational number nearby. Amazingly, after several thousands of years of playing around with rational numbers, some mathematicians of the ancient world discovered something deeply interesting about this drawing: *there are still gaps*. Indeed, there are special numbers like $\sqrt{2}$ that seem to exist, yet *cannot* be written as a ratio $\frac{a}{b}$. Such numbers are called *irrational*. If we wanted to take the diagram above and "fill in the gaps" by including all of the irrational numbers as well, then the result would be a continuous line:



This continuous line should be understood as a collection of numbers: any point on this line corresponds to an individual number, which could be a natural number, perhaps an integer, or a rational number or an irrational number. The collection of all of these is known as the *real numbers*, and we write it with the letter $\mathbb{R}$.

The arrangement of all the numbers into a line allows us to think of numbers as *ordered*. We usually write this ordering with the symbol $<$, i.e. writing $a < b$ means that the number a is less than the number b. Geometrically, this means that a sits to the left of b on the number line. The following summarises our discussion so far.

Real Numbers

The set of real numbers, written $\mathbb{R}$, contains all the numbers that can be placed on the number line. Important types of real numbers include:

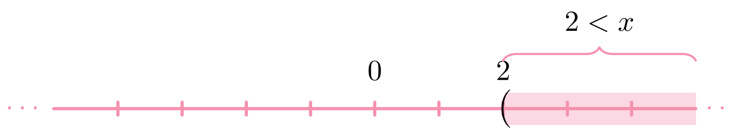
- **Natural numbers:** $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$.
- **Integers:** $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- **Rational numbers:** numbers that can be written as fractions $\dfrac{a}{b}$, where a, b in $\mathbb{Z}$ and $b \neq 0$.
- **Irrational numbers:** real numbers that cannot be written as fractions, such as $\sqrt{2}$, π, and e.

1.1 Inequalities

The number line gives us a useful way to understand inequalities. For example, the inequality

$$2 < x$$

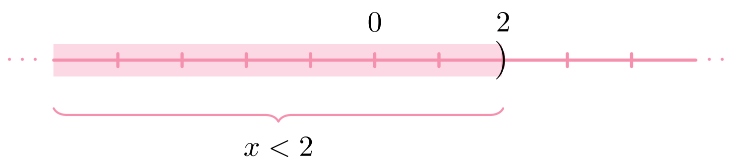
means that x is a real number to the right of 2. Equivalently, we could write this as $x > 2$.



Similarly, the inequality

$$x < 2$$

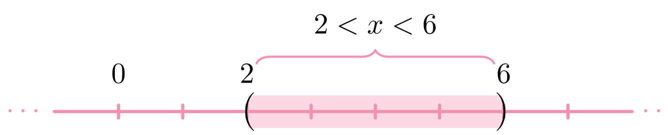
means that x is a real number to the left of 2.



We can also describe numbers between two endpoints. For example,

$$2 < x < 6$$

means that x is larger than 2 but smaller than 6.



Inequalities and Intervals

The following interpretations are useful:

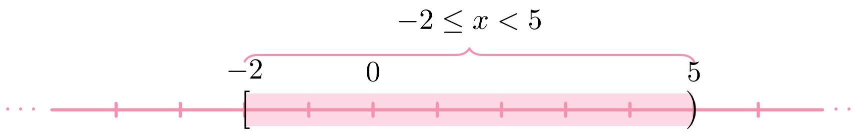
- $x < a$ means values to the left of a.
- $x > a$ means values to the right of a.
- $a < x < b$ means values between a and b.
- A square bracket means that an endpoint is included.
- A round bracket means that an endpoint is not included.

For example, all real numbers between -2 and 5, including -2 but not 5, can be written as

$$[-2, 5).$$

The same interval can also be described by the inequality

$$-2 \leq x < 5.$$



1.2 Absolute Values

Before getting to the "big five" operations, we will start with a brief discussion of the absolute value operation. In very simple terms, the absolute value is an operation that removes the minus sign in front of a number. We write the operation with two vertical bars, for example

$$|-3| = 3, \quad \text{and} \quad |-8| = 8.$$

In the situation that there is no minus sign to remove, then the absolute value operation does absolutely nothing, for example

$$|2| = 2, \quad \text{and} \quad |15| = 15.$$

As a matter of fact, the absolute value operation has a geometric meaning on our number line: it gives us a way to express *distances*.

Absolute Value as a Distance

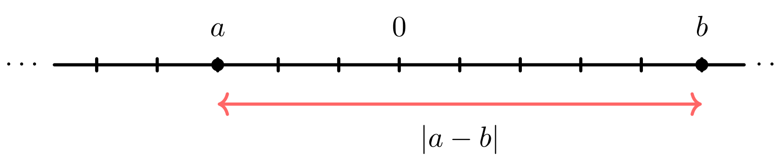
For any number a in $\mathbb{R}$, the absolute value $|a|$ gives the distance from a to 0 on the number line. Since distances are never negative, we always have

$$|a| \geq 0.$$

We may also use this idea to calculate the distances between different real numbers. The distance between two numbers a and b is given by

$$|a - b|.$$

Graphically, the operation $|a - b|$ is doing something like the following:



Exercise 1

Compute the following.

(1) $|-3|$
(2) $|7|$
(3) $-|4|$
(4) $-|-1|$
(5) $|0|$
(6) $-|3-5|$

2 Arithmetic Operations

We will now set about describing some arithmetic operations on the real numbers. In reality, there are *many* things you can do with the real number line. For now, we will introduce the five basic operations. These are:

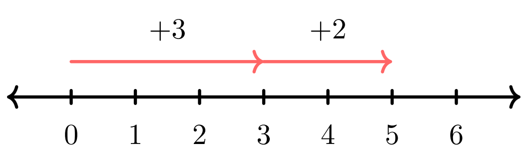
(1) Addition: combining quantities or changes.
(2) Subtraction: difference, or adding the opposite.
(3) Multiplication: repeated addition or scaling.
(4) Division: splitting into groups, or multiplying by the reciprocal.
(5) Exponentiation: repeated multiplication.

2.1 Addition

Addition combines two numbers to make a third number. Of course, we may understand addition in terms of combinations of quantities, for example $3 + 2 = 5$ can be understood as:

$$\bullet \;\bullet\; \bullet \quad + \quad \bullet\; \bullet \quad = \quad \bullet\; \bullet\; \bullet\; \bullet\; \bullet$$

However, on the number line we may also interpret 3 and 2 as continuous *lengths*. That means addition would combine these lines together to create a larger line of length 5:



Using either of these interpretations, we can discover a collection of basic properties of addition that hold for all real numbers. Below are four of these important properties.

Properties of Addition

For all real numbers a, b, c in $\mathbb{R}$:

- **Commutative property:**
$$a + b = b + a.$$
For example, $3 + 5 = 5 + 3$.
- **Associative property:**
$$(a + b) + c = a + (b + c).$$
For example, $(2 + 6) + 5 = 2 + (6 + 5)$.
- **Additive identity:**
$$a + 0 = 0 + a = a.$$
- **Additive inverse:**
$$a + (-a) = 0.$$

Recall that for large nonnegative integers, we can add vertically using the carrying algorithm. For example:

$$\begin{array}{cccc} & {\scriptstyle 1} & {\scriptstyle 1} & \\ & 1 & 4 & 8 \\ & & 6 & 2 \\ + & 5 & 3 & 6 \\ \hline & 7 & 4 & 6 \end{array} \qquad \text{so} \qquad 148 + 62 + 536 = 746.$$

Here we start in the right-most column and add those numbers. If the answer exceeds 10, then we add an extra 1 into the next column and repeat the procedure until we run out of columns.

2.2 Subtraction

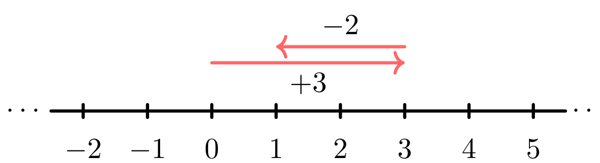
Subtraction is really addition of the opposite:

$$3 - 2 = 3 + (-2).$$

For this reason, subtraction is closely related to the additive inverse property: for any real number a, subtracting a from a quantity is the same as adding the number $-a$ to it. In this sense, subtraction and addition are opposite operations. In terms of quantities, subtraction is simply the act of *removing* things. For instance, $3 - 2 = 1$ is:

$$\bullet \; \bullet \; \bullet \quad - \quad \bullet \; \bullet \quad = \quad \bullet$$

If we instead choose to interpret numbers as continuous lengths, then subtraction is the removal of a piece of line, or equivalently, adding a line in the opposite direction:



For subtraction problems involving two nonnegative integers, we can use the borrowing algorithm. For example:

$$\begin{array}{cccc} & 1 & 3 & 17 \\ & 1 & \cancel{4} & \cancel{7} \\ - & & 2 & 9 \\ \hline & 1 & 1 & 8 \end{array} \qquad \text{so} \qquad 147 - 29 = 118.$$

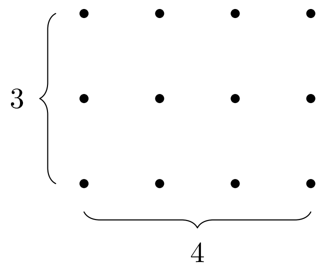
The idea here is that the 7 in the units column is too small to subtract 9, so we "borrow" 1 ten from the tens column. This turns 47 into 3 tens and 17 ones.

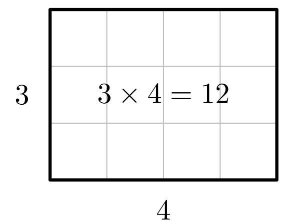
2.3 Multiplication

For positive integers a and b, multiplication is simply just repeated applications of addition:

$$a \times b = \underbrace{b + b + \cdots + b}_{a\text{-many times}}.$$

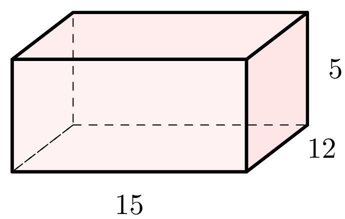
We often write multiplication as either $a \times b$ or $a \cdot b$. Multiplication also has a geometric meaning: if length is represented by a line, then multiplication of two lengths gives a two-dimensional *area*.





Multiplication of three lengths gives a volume. For instance, a rectangular box with length 15, width 12, and height 5 has volume

$$15 \times 12 \times 5 = 900.$$



As with addition, multiplication has some important properties.

Properties of Multiplication

For all real numbers a, b, c in $\mathbb{R}$:

- **Commutative property:**
$$ab = ba.$$
For example, $4 \cdot (-7) = (-7) \cdot 4$.
- **Associative property:**
$$(ab)c = a(bc).$$
For example, $(3 \cdot 5) \cdot 2 = 3 \cdot (5 \cdot 2)$.
- **Multiplicative identity:**
$$a \cdot 1 = 1 \cdot a = a.$$
- **Multiplicative inverse:**
$$a \cdot a^{-1} = a^{-1} \cdot a = 1, \qquad a \neq 0.$$
Equivalently,
$$a \cdot \frac{1}{a} = 1, \qquad a \neq 0.$$

For large integers, we can multiply using another of these vertical algorithms. For example:

$$\begin{array}{rrrr} & & 2 & 3 \\ \times & & 1 & 4 \\ \hline & & 9 & 2 \\ & 2 & 3 & 0 \\ \hline & 3 & 2 & 2 \end{array} \qquad \text{so} \qquad 23 \times 14 = 322.$$

We will practice this more in the exercise classes.

2.4 Division

In the same sense that subtraction is really just another form of addition, division is really just another form of multiplication. This time using the multiplicative inverse property above, dividing by a nonzero number is the same as multiplying by its reciprocal:

$$8 \div 2 = 8 \times 2^{-1} = 8 \times \frac{1}{2} = 4.$$

Division Vocabulary

The result of dividing one number by a nonzero number is called the **quotient**. The division

$$30 \div 6$$

may also be written as

$$30/6 \qquad \text{or} \qquad \frac{30}{6}.$$

In the expression $30 \div 6$, the number 30 is the **dividend** and 6 is the **divisor**. In the fraction $\frac{30}{6}$, the number 30 is the **numerator** and 6 is the **denominator**.

Some important rules for division are:

- Zero divided by a nonzero integer is 0.
- A nonzero integer divided by zero is undefined.
- The quotient of two nonzero integers with the same sign is positive.
- The quotient of two nonzero integers with different signs is negative.

For example:

$$\frac{-42}{-6} = 7, \qquad 36 \div (-9) = -4, \qquad 0 \div (-13) = 0, \qquad -97 \div 0 \text{ is undefined}.$$

For large numbers, we can use long division. For instance:

$$\begin{array}{rr|rrr} & & 2 & 7 & \\ \cline{3-5} 1 & 3 & 3 & 5 & 1 \\ & & 2 & 6 & \\ \cline{3-4} & & & 9 & 1 \\ & & & 9 & 1 \\ \cline{4-5} & & & & 0 \end{array} \qquad \text{so} \qquad 351 \div 13 = 27.$$

2.5 Factors and Prime Numbers

Factors and Prime Numbers

Let a and b be positive integers. We say that a is a **factor** or **divisor** of b if a divides evenly into b.
For example, $1, 2, 3$, and 6 are factors of 6.
Positive integers can be classified as follows:

- The number 1 is neither prime nor composite.
- A **prime number** is an integer greater than 1 whose only positive factors are 1 and itself.
- A **composite number** is an integer greater than 1 with more than two positive factors.

Every composite number can be written as a product of prime numbers. This is called its **prime decomposition** or **prime factorisation**. For example:

$$84 = 2 \cdot 42 = 2 \cdot 2 \cdot 21 = 2 \cdot 2 \cdot 3 \cdot 7.$$

Similarly,

$$78 = 2 \cdot 39 = 2 \cdot 3 \cdot 13.$$

The prime decomposition of an integer greater than 1 is always unique up to order, and therefore in a way it characterises the number. This is why we call prime numbers "prime" – they can be used to build all integers greater than 1 by taking products of them.

Exercise 2

Write the prime factorisation of each number.

(1) 68 **(2)** 133 **(3)** 43

2.6 The Distributive Property

The distributive property explains how multiplication interacts with addition:

$$a(b + c) = ab + ac.$$

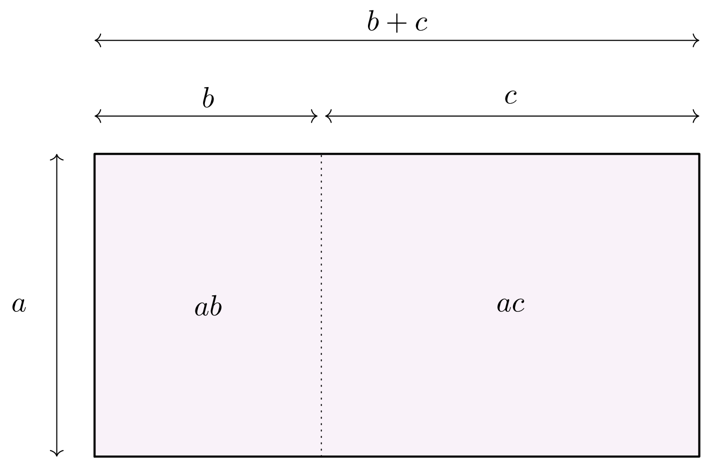
Equally,

$$(a + b)c = ac + bc.$$

For example:

$$3(8 + 5) = 3 \cdot 8 + 3 \cdot 5.$$

The distributive property has a geometric interpretation. The area of one large rectangle can be split into the areas of two smaller rectangles.



The total area can be computed in two equivalent ways:

$$\text{Total area} = a(b+c)$$

or

$$\text{Total area} = ab + ac.$$

Therefore:

$$a(b+c) = ab + ac.$$

2.7 Exponentiation

Multiplication is repeated addition. For a positive integer b, exponentiation is repeated multiplication:

$$a^b = \underbrace{a \cdot a \cdot \ldots \cdot a}_{b\text{-many times}}.$$

Here a is called the **base**, and b is called the **exponent**.

For example:

$$2^3 = 2 \cdot 2 \cdot 2 = 8.$$

Rules of Exponents

For nonzero real numbers a and b, and integers m and n, we have:

$$\begin{aligned} a^m a^n &= a^{m+n}, \\ \frac{a^m}{a^n} &= a^{m-n}, \\ (a^m)^n &= a^{mn}, \\ (ab)^n &= a^n b^n. \end{aligned}$$

For example:

$$2^3 \cdot 2^2 = 2^{3+2} = 2^5 = 32.$$

We will discuss these rules at length in the later lectures. For now, it is enough to simply remember that they exist.

2.8 Order of Operations

When an expression contains several operations, we need a standard order for evaluating it.

Order of Operations

The usual order of operations is:

(1) **P**arentheses first,

(2) then **E**xponents,

(3) then **M**ultiplication and **D**ivision,

(4) and finally **A**ddition and **S**ubtraction.

The abbreviation PEMDAS is commonly used.

As an example, suppose we have some real number a, and write something like:

$$3(a-7)^2+2.$$

According to our order of operations, this means:

subtract 7, then square the result, then multiply by 3, then add 2.

You should be careful when manipulating negative numbers and brackets. For instance, observe that:

$$(-3)^2=(-3)(-3)=9,$$

but

$$-3^2=-(3^2)=-9,$$

since in the latter, the exponent comes *first* before the minus sign.

Exercise 3

Evaluate the following.

(1) -3^2

(2) $(-3)^2$

(3) $5+2^2$

(4) $(5+2)^2$

(5) $3\cdot 2^2$

(6) $(3\cdot 2)^2$

3 Fractions

A fraction

$$\frac{a}{b}$$

means $a\div b$, where $b\neq 0$. Fractions can represent:

- a part of a whole,
- a ratio,
- a rate, such as "per".

For example:

$$\frac{3}{4}=0.75.$$

3.1 Simplifying Fractions

To simplify a fraction, divide the numerator and denominator by the same nonzero number. The goal is to write the fraction in simplest form, meaning that the numerator and denominator have no common factor besides 1.

For example:

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}.$$

One method is to use prime decomposition:

$$\frac{25}{100} = \frac{5 \cdot 5}{2 \cdot 2 \cdot 5 \cdot 5} = \frac{1}{2 \cdot 2} = \frac{1}{4}.$$

Another method is to use the greatest common factor. The greatest common factor of two positive integers is the largest number that divides both of them. For example, the greatest common factor of 14 and 21 is 7, so

$$\frac{14}{21} = \frac{2 \cdot 7}{3 \cdot 7} = \frac{2}{3}.$$

3.2 Adding and Subtracting Fractions

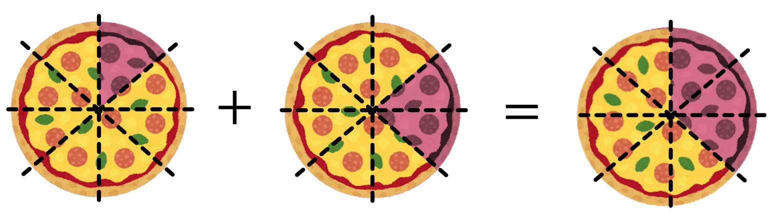
To add or subtract fractions, we need a common denominator which is nonzero:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

If the denominators are different, rewrite the fractions using a common denominator. For example:

$$\frac{1}{8} + \frac{1}{4} = \frac{1}{8} + \frac{2}{8} = \frac{3}{8},$$

or, visually:



As you can see, we made a subdivision of the second pizza in order to represent the quarter in terms of eights. More generally, where $c, d \neq 0$:

$$\frac{a}{c} + \frac{b}{d} = \frac{ad}{cd} + \frac{bc}{cd} = \frac{ad+bc}{cd}.$$

3.3 Multiplying Fractions

To multiply fractions, multiply the numerators and multiply the denominators where the denominators are nonzero:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

For example:

$$\frac{3}{5} \cdot \frac{10}{7} = \frac{30}{35} = \frac{6}{7}.$$

3.4 Dividing Fractions

Dividing by a nonzero number is multiplying by its reciprocal:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, \qquad b, c, d \neq 0.$$

For example:

$$\frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \cdot \frac{6}{5} = \frac{12}{15} = \frac{4}{5}.$$

Exercise 4

Compute the following.

(1) $\frac{5}{2} + \frac{3}{4}$

(2) $\frac{5}{2} \cdot \frac{3}{4}$

(3) $\frac{6}{3} \div \frac{1}{2}$

(4) $\frac{18}{24}$

(5) $\frac{7}{8} - \frac{1}{4}$

(6) $\frac{3}{5} \cdot \frac{15}{9}$

4 Decimals

4.1 Place value

Digits to the right of the decimal point represent tenths, hundredths, thousandths and so on. For example,

$$0.4 = \frac{4}{10}, \qquad 0.04 = \frac{4}{100}.$$

Similarly,

$$2.305 = 2 + \frac{3}{10} + \frac{0}{100} + \frac{5}{1000}.$$

Number	Ones	Tenths	Hundredths	Thousandths
2.305	2	3	0	5

4.2 Converting fractions and decimals

If the denominator is $10, 100, 1000, \ldots$, then converting to a decimal is immediate. For example,

$$\frac{7}{100} = 0.07.$$

Otherwise, division converts a fraction to a decimal. For example,

$$\frac{3}{4} = 0.75.$$

The important point is that fractions and decimals are often two different ways of writing the same number. For example,

$$\frac{1}{2} = 0.5, \qquad \frac{1}{4} = 0.25, \qquad \frac{3}{4} = 0.75.$$

4.3 Rounding decimals

Rounding Decimals

To round a decimal:

- if the next digit is less than 5, round down;
- if the next digit is 5 or greater, round up.

The phrase "round to n decimal places" means that the answer should have n digits after the decimal point.

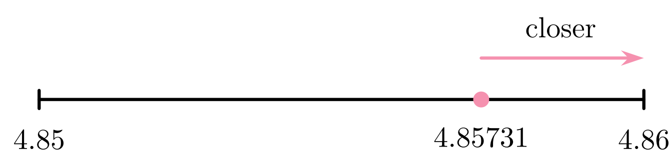
For example, consider the number

$$4.85731.$$

Then:

- rounded to 2 decimal places:
$$4.85731 \approx 4.86;$$
- rounded to 3 decimal places:
$$4.85731 \approx 4.857;$$
- rounded to 4 decimal places:
$$4.85731 \approx 4.8573.$$

Another way to understand rounding is that we are choosing which nearby number is closer. For example, when rounding 4.85731 to 2 decimal places, we are comparing it to 4.85 and 4.86.



4.4 Decimal arithmetic

Decimals can be added, subtracted, multiplied and divided using the same vertical algorithms as before. The main extra detail is to keep careful track of the decimal point.

For example,

$$0.58 + 1.06 = 1.64.$$

Multiplying by 10 shifts the decimal point one place to the right, while dividing by 10 shifts it one place to the left. For example,

$$4.56 \times 100 = 456.$$

A good habit is to estimate before computing. For instance,

$$19.8 \times 3.1$$

should be close to

$$20 \times 3 = 60.$$

This tells us that if our final answer is wildly different from 60, we should check our work.

Exercise: Decimals

Compute the following.

(a) $0.58 + 1.06$

(b) Convert $\frac{7}{100}$ to a decimal.

(c) Convert $\frac{3}{4}$ to a decimal.

(d) Round 4.85731 to 2 decimal places.

(e) Round 4.85731 to 3 decimal places.

(f) Round 4.85731 to 4 decimal places.

Solution

(a) $0.58 + 1.06 = 1.64$.

(b) $\frac{7}{100} = 0.07$.

(c) $\frac{3}{4} = 0.75$.

(d) $4.85731 \approx 4.86$.

(e) $4.85731 \approx 4.857$.

(f) $4.85731 \approx 4.8573$.

5 Percentages

5.1 What is a percentage?

The word "percent" literally means "per hundred". The word "cent" means 100, as in the word "century". So, a percent is a number of parts per one hundred.

For example,

$$25\% = 0.25 = \frac{25}{100}.$$

Similarly,

$$100\% = 1.00 = \frac{100}{100}.$$

Percentages

A percentage is a way of writing a number as a quantity out of 100. For example,

$$37\% = \frac{37}{100} = 0.37.$$

5.2 Converting between percentages, decimals and fractions

To convert a percentage to decimal form, move the decimal point two places to the left. For example,

$$100\% = 1.00, \qquad 25\% = 0.25, \qquad 3.5\% = 0.035.$$

To convert a decimal to a percentage, move the decimal point two places to the right. For example,

$$1.00 = 100\%, \qquad 1.20 = 120\%, \qquad 0.005 = 0.5\%.$$

Exercise: Converting to Percents

Convert the following to percents.

(a) 1.20

(b) $\frac{3}{5}$

Solution

(a)

$$1.20 = \frac{120}{100} = 120\%.$$

(b)

$$\frac{3}{5} = \frac{3 \cdot 20}{5 \cdot 20} = \frac{60}{100} = 60\%.$$

Exercise: Converting from Percents

(a) Convert 3.5% to a decimal.

(b) Convert 55% to a fraction.

Solution

(a)

$$3.5\% = \frac{3.5}{100} = \frac{35}{1000} = 0.035.$$

(b)

$$55\% = \frac{55}{100} = \frac{5 \cdot 11}{5 \cdot 20} = \frac{11}{20}.$$

5.3 Percentage equations

The primary use of percentages is to compare two numbers. For example, 2 is 50% of 4, and 5 is 25% of 20.

The following model is helpful:

$$a = p \text{ percent of } b.$$

Here:

- b is the base number;
- p is the percent, written in decimal form;
- a is the number being compared to b.

In equation form, this becomes

$$a = p \cdot b.$$

Exercise: Percentage Equation

What number is 30% of 70?

Solution

We use the equation

$$a = p \cdot b.$$

Here $p = 0.30$ and $b = 70$, so

$$a = (0.30)(70) = 21.$$

Therefore, 21 is 30% of 70.

5.4 A neat percentage trick

When asking "what is p% of b?", we are calculating

$$\frac{p}{100} \cdot b.$$

But multiplication is commutative, so

$$\frac{p}{100} \cdot b = \frac{pb}{100} = \frac{bp}{100} = \frac{b}{100} \cdot p.$$

Therefore,

$$p\% \text{ of } b = b\% \text{ of } p.$$

The Percentage Symmetry Trick

For many mental calculations, it is useful to remember:

$$p\% \text{ of } b = b\% \text{ of } p.$$

This does not change the mathematics. It only changes which calculation is easier to do in your head.

For example:

- 7% of 100 is the same as 100% of 7, which is 7.
- 21% of 50 is the same as 50% of 21, which is 10.5.
- 63% of 90 is the same as 90% of 63, which is $63 - 6.3 = 56.7$.

Exercise: Arithmetic Review

Compute the following.

(1) $15 + 3$

(2) $12 - 4$

(3) $-7 + 3$

(4) $|-3 - 4|$

(5) 7×3

(6) $14 \div 2$

(7) $\frac{1}{3} + \frac{1}{4}$

(8) $\frac{2}{3} \cdot \frac{1}{2}$

(9) $\frac{2}{3} \div \frac{1}{2}$

(10) $0.07 = \frac{?}{100}$

(11) $0.28 = \frac{?}{50}$

(12) 15% of 60

Solution

(1)
$$15 + 3 = 18.$$

(2)
$$12 - 4 = 8.$$

(3)
$$-7 + 3 = -4.$$

(4)
$$|-3 - 4| = |-7| = 7.$$

(5)
$$7 \times 3 = 21.$$

(6)
$$14 \div 2 = 7.$$

(7)
$$\frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}.$$

(8)
$$\frac{2}{3} \cdot \frac{1}{2} = \frac{2}{6} = \frac{1}{3}.$$

(9)
$$\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \cdot \frac{2}{1} = \frac{4}{3}.$$

(10)
$$0.07 = \frac{7}{100}.$$

(11) Since
$$0.28 = \frac{28}{100} = \frac{14}{50},$$
we have
$$0.28 = \frac{14}{50}.$$

(12)
$$15\% \text{ of } 60 = 0.15 \cdot 60 = 9.$$