MAT120: Lecture 7 Handout
Linear Models

Today we will start our next topic: mathematical modelling. The idea of modelling is that we can use mathematics to describe various things in the real world. These descriptions are only ever approximations, but usually they are “good enough” to be helpful and insightful. Today we will explore the simplest of these mathematical models, which are the linear models. As the name suggests, these models use straight lines to model systems that have a constant change. We will start by exploring the mathematics of straight lines, and then we will get to solving some small problems.

1 Linear Equations

1.1 Linear Equations

Linear equations are the simplest type of algebraic equation. A linear equation in one variable x is an equation that can be written in the standard form

$$mx + b = 0.$$

Here the letters m and b are fixed numbers, and x is a variable (i.e. it is allowed to take on many values). We also assume that m is non-zero, since otherwise our equation would read $b = 0$, which is not particularly interesting.

Some examples of linear equations in standard form are:

(1) $2x = 0$ here $m = 2$ and $b = 0$.
(2) $x - 7 = 0$ here $m = 1$ and $b = -7$.
(3) $4x + 6 = 0$ here $m = 4$ and $b = 6$.
(4) $\frac{x}{2} - 1 = 0$ here $m = \frac{1}{2}$ and $b = -1$.

1.2 Solving Linear Equations in Standard Form

Remember that to solve an equation involving x means to find all values of x that make the equation true. For a linear equation of the form $mx + b = 0$, the goal is to isolate x by rewriting the equation so that:

$$x = \text{a number}.$$

How do we do this? We start with the original equation and write a sequence of equivalent equations, each having the same solution as the original equation. In order to change the equation and keep the balance the same, you must do the same thing to both sides of the equation. For instance, to solve the linear equation $x - 2 = 0$, you can add 2 to both sides of the equation to obtain $x = 2$.

1.3 Worked Exercises

It is often best to simply see solutions in action. So, we will now solve several linear equations. To start with, consider the equation $3x - 15 = 0$. We can solve this by first adding +15 and then dividing everything by 3. The derivation is written line-by-line:

$$3x - 15 = 0$$
$$3x - 15 + 15 = 0 + 15 \qquad \text{(add 15 to both sides)}$$
$$3x = 15 \qquad \text{(simplify)}$$
$$x = 5 \qquad \text{(divide both sides by 3)}$$

It appears that the solution is $x = 5$. We can double-check this by substituting the value $x = 5$ back into the original equation:

$$3(5) - 15 = 15 - 15 = 0,$$

so our solution is indeed correct.

Exercise 1

Solve the equation $4x + 12 = 4$.

Solution

$$\begin{aligned} 4x + 12 &= 4 \\ 4x + 12 - 12 &= 4 - 12 \qquad \text{(subtract 12 from both sides)} \\ 4x &= -8 \\ x &= -2 \qquad \text{(divide both sides by 4)} \end{aligned}$$

To double-check: $4(-2) + 12 = -8 + 12 = 4$, which is correct.

Exercise 2

Solve the equation $\frac{z}{5} + 1 = 6$. Then check the solution.

Solution

$$\begin{aligned} \frac{z}{5} + 1 &= 6 \\ \frac{z}{5} + 1 - 1 &= 6 - 1 \\ \frac{z}{5} &= 5 \\ z &= 25 \end{aligned}$$

To double-check: $\frac{25}{5} + 1 = 5 + 1 = 6$.

Exercise 3

Solve the equation $5(x + 2) = 2(x - 1)$.

Solution

$$\begin{aligned} 5(x + 2) &= 2(x - 1) \\ 5x + 10 &= 2x - 2 \qquad \text{(distributive property)} \\ 5x - 2x + 10 &= 2x - 2x - 2 \qquad \text{(subtract } 2x \text{ from both sides)} \\ 3x + 10 &= -2 \\ 3x + 10 - 10 &= -2 - 10 \qquad \text{(subtract 10 from both sides)} \\ 3x &= -12 \\ x &= -4 \end{aligned}$$

To double-check: substituting $x = -4$ into the left hand expression gives $5(-4 + 2) = 5(-2) = -10$, and substituting $x = -4$ into the right hand expression gives $2(-4 - 1) = 2(-5) = -10$. Both sides are equal, therefore the solution $x = -4$ is correct.

2 Graphs of Linear Equations

2.1 Ordered Pairs as Solutions

An *ordered pair* is a pair of real numbers (x, y) where the order matters, meaning that $(x, y) \neq (y, x)$. Generally:

$$(a, b) = (c, d) \text{ if and only if } a = c \text{ and } b = d.$$

Suppose now that you have an equation with two variables:

$$y + 2x = 3.$$

Solutions for equations like this will require not just a value of x, but also a value of y. We may represent these two values as an ordered pair (x, y), so that a solution will be a pair of numbers instead of just one number. For example, for the equation above:

- The pair $(1, 1)$ is a solution because $1 + 2 \cdot 1 = 3$.
- The pair $(2, 1)$ is *not* a solution because $1 + 2 \cdot 2 = 5 \neq 3$.
- The pair $(0, 3)$ is a solution because $3 + 2 \cdot 0 = 3$.

Exercise 4

Consider the equation $y + x^2 = 4$. Determine if the following ordered pairs are solutions or not.

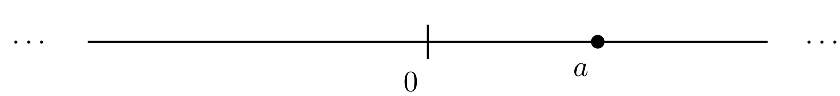
(a) $(1, 2)$
(b) $(0, -2)$
(c) $(3, -5)$
(d) $(0, 4)$

Solution

(a) $(1, 2)$: $2 + 1^2 = 2 + 1 = 3 \neq 4$ therefore $(1, 2)$ is not a solution.
(b) $(0, -2)$: $-2 + 0^2 = -2 + 0 = -2 \neq 4$ therefore $(0, -2)$ is not a solution.
(c) $(3, -5)$: $-5 + 3^2 = -5 + 9 = 4$ therefore $(3, -5)$ is a solution.
(d) $(0, 4)$: $4 + 0^2 = 4 + 0 = 4$, therefore $(0, 4)$ is a solution.

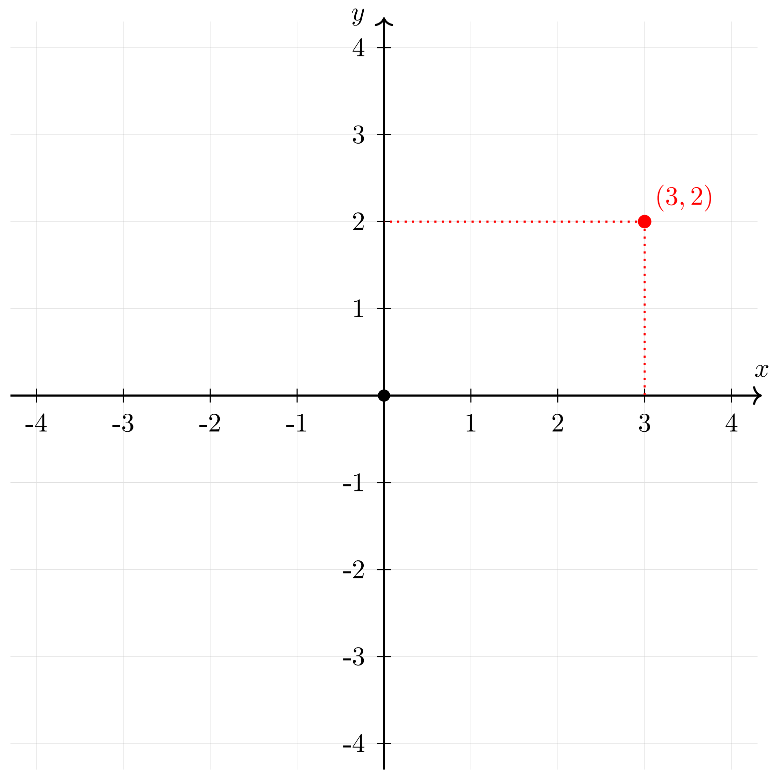
2.2 The Rectangular Coordinate System

Recall that the real line is the collection of all the real numbers, arranged in a line according to the ordering of $<$.[1] Moreover, any real number a can be represented as a point on this line:



[1]Writing $a < b$ meant that the number a was to the left of the number b on the real line.

Similarly, there is also a real *plane*, which is a two-dimensional grid that represents all the possible ordered pairs (x, y), where x and y are real numbers. We often denote the real number line by $\mathbb{R}$, and therefore we call the real number plane $\mathbb{R}^2$. Another name for $\mathbb{R}^2$ is a *coordinate chart*. The diagram below depicts $\mathbb{R}^2$, with one labelled point.



On the real number plane $\mathbb{R}^2$, there are two important lines. These are called the x-axis and the y-axis, and they act as a way to navigate the plane. You can imagine the x-axis as a copy of the number line: a horizontal line that is infinitely long. On the x-axis there is 0, also called the origin. From here, we take another copy of the real line, rotate it by 90° so that it is vertical, and then attach it to the x-axis at the origin. The y-axis is this vertical line. Together, the two lines make a cross-shape, and they meet in the middle at the coordinates $(0, 0)$.

The letter x denotes a variable, i.e. a number that is allowed to change its value. We can imagine that as x changes its value, it slides around on the horizontal x-axis. Similarly, y is also a variable, so it is a number that is allowed to change its value. We can imagine that when y changes its value, it slides up and down the vertical y-axis.

Using these two sliders, we can represent any point in two dimensions: we first move horizontally and then we move vertically. Each point in $\mathbb{R}^2$ can be described using two numbers: an x-coordinate and a y-coordinate. This is precisely an ordered pair like (x, y).

For example, the ordered pair $(3, 2)$ represents a point that is 3 moves to the right of the origin and then 2 moves up. If we have a negative x component then we move to the left, and if we have a negative y component then we move down the grid.

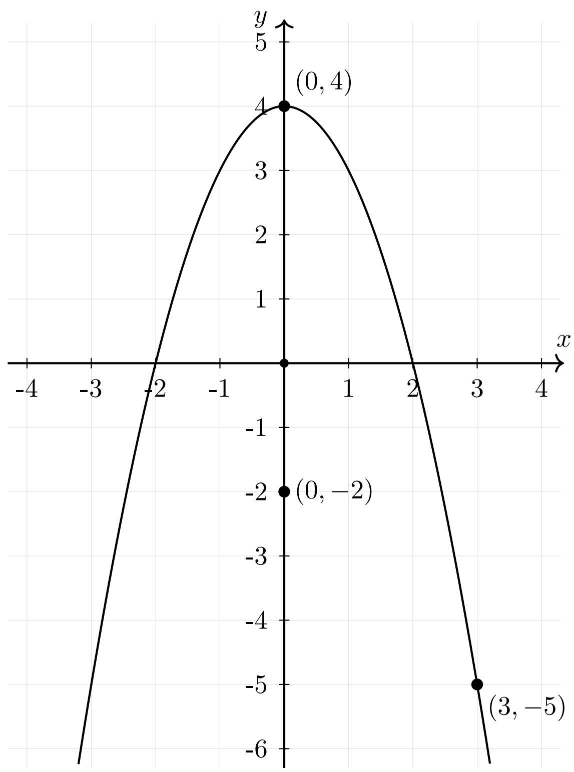
2.3 Ordered Pairs and Graphs

We saw earlier that ordered pairs can represent solutions to equations that involve both x and y variables. Interestingly, we can represent the collection of all solutions to an equation using a picture. This picture is called the *graph of the equation*, and it is defined to be the set of all ordered pairs that are a solution to the equation.

There are two ways that a graph is useful:

1. We can use a graph to visually represent the relationship between y and x.
2. We can use a graph to read off solutions without needing to do any algebra.

As an example, we now consider the equation $y + x^2 = 4$. Observe first that we can rewrite this equation as $y = 4 - x^2$. A sketch of the coordinate chart and the graph is shown below, including some labelled points.



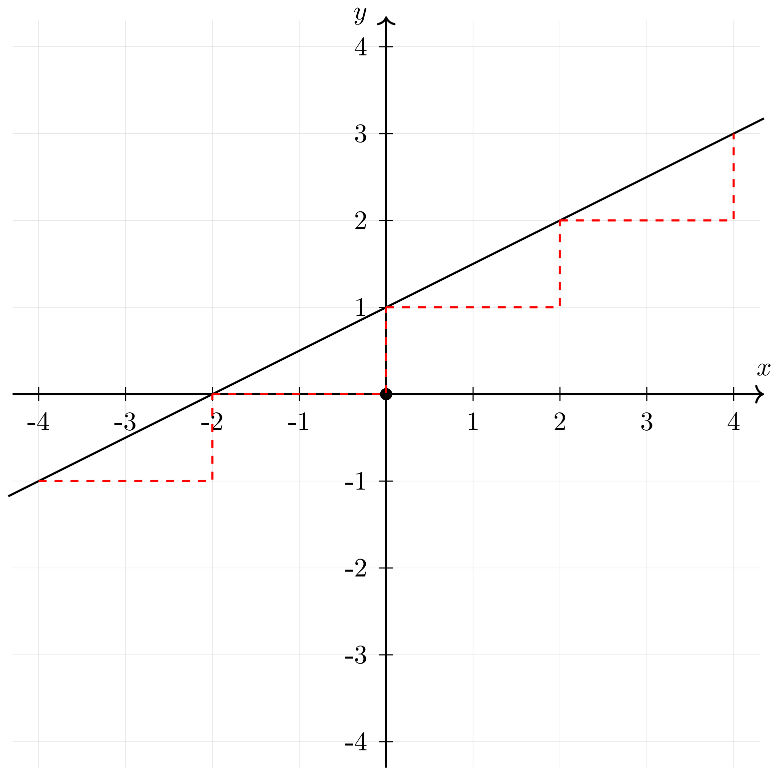
Notice that there are two labelled points $(0, 4)$ and $(3, -5)$ that sit on the graph, and labelled point $(0, -2)$ that does not. If we substitute these into the equation $y = 4 - x^2$, we see that:

- $4 = 4 - 0^2$,
- $-5 = 4 - (3)^2$, and
- $-2 \neq 4 - 0^2$.

Therefore, the first two points sitting on the graph are solutions to the equation $y = 4 - x^2$, whereas the third point $(0, -2)$ is not a solution, which is why it is not on the graph.

2.4 Gradient

Since we are ultimately concerned with *linear* models, for the rest of this lecture we will focus only on straight lines. These graphs are particularly useful because they model a constant growth of y in terms of x, that is, the line changes at a steady rate:



In the diagram above, notice how the line changes at a constant rate: for every 2 squares we move to the right, the line always moves up 1 square. The gradient of a line captures this idea: it is simply a number which tells us how steep the line is, i.e. how much it is changing vertically when we move around horizontally. So, a line that is more steep will have a larger gradient, and a line that is less steep will have a smaller gradient.

We can describe the gradient of a straight line by comparing the change in vertical direction to the change in horizontal direction. This comparison is simply a ratio:

$$\text{Gradient} = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\text{Vertical change}}{\text{Horizontal change}}.$$

We can be more precise by picking two points (x_1, y_1) and (x_2, y_2) that are on the line. Then, the gradient, written m, can be calculated using the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{where } x_1 \neq x_2.$$

Why do we need $x_1 \neq x_2$? Because if they were the same, then $x_2 - x_1 = 0$ and therefore we would be dividing by zero, which is not possible. However, it is completely fine for $y_1 = y_2$, since that corresponds to a zero gradient. Also, note that since the line is changing at a constant rate, it doesn't matter which two points we sample from the line.

When the formula for the gradient is used, it is important to keep track of the order of subtractions. Given two points on a line, you are free to label either of them (x_1, y_1) and the other (x_2, y_2). However, once this has been done, you must form the numerator and denominator using the same order of subtraction. Two correct formulas are:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{and} \qquad m = \frac{y_1 - y_2}{x_1 - x_2}$$

whereas two **incorrect** formulas are:

$$m = \frac{y_2 - y_1}{x_1 - x_2} \qquad \text{and} \qquad m = \frac{y_1 - y_2}{x_2 - x_1}.$$

Exercise 5

Find the gradient of the straight line passing through each pair of points.

(a) The points $(-2, 0)$ and $(3, 1)$.

(b) The points $(0, 0)$ and $(1, -1)$.

Solution

(a) Let $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (3, 1)$. Then

$$m = \frac{1 - 0}{3 - (-2)} = \frac{1}{5}.$$

(b) Using $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, -1)$:

$$m = \frac{-1 - 0}{1 - 0} = -1.$$

2.5 Gradient Visualized

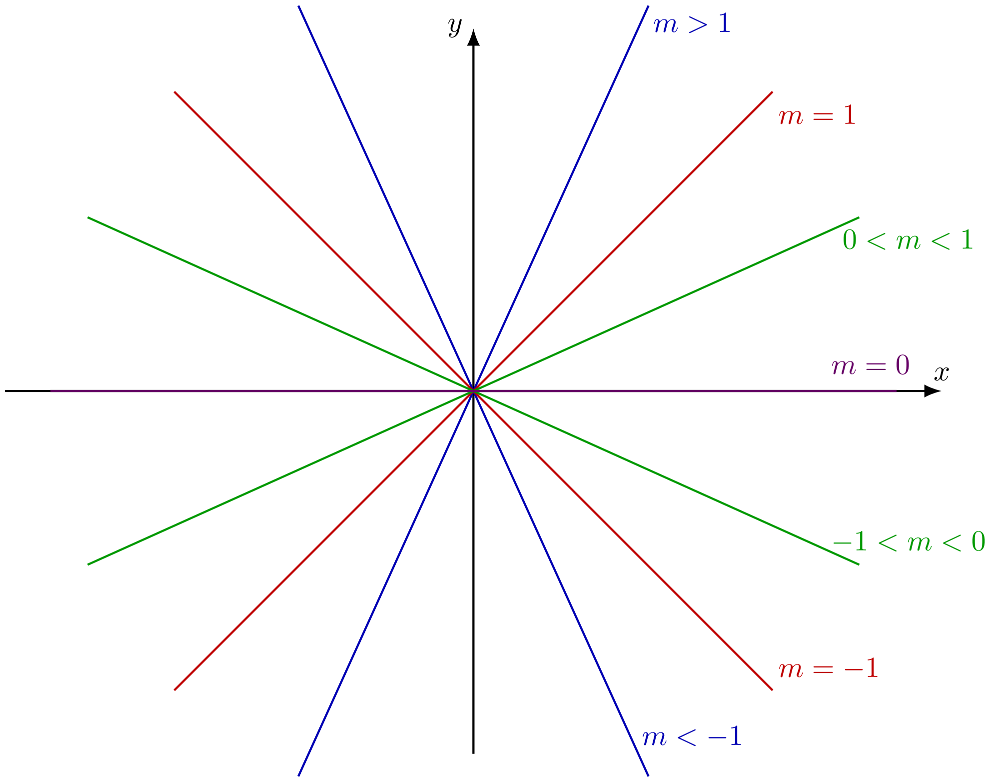
Imagine that you have a straight line with gradient $m = 1$. According to the formula for gradient, this means that the ratio of vertical to horizontal change is precisely 1. For example, if we move 1 unit in the horizontal direction, then the line also moves up 1 unit in the vertical direction. Similarly, if we move 2 units horizontally then the line moves up 2 units.

We can imagine a situation in which the gradient is larger than 1. In this situation, the line will be steeper than 45°, meaning that it moves up more in the vertical direction than it does in the horizontal direction.

We can also imagine the situation in which a line moves more in the horizontal direction than it does in the vertical direction. In this case, the gradient would satisfy $0 < m < 1$.

If there is no vertical change at all, then $y_2 - y_1 = 0$ and therefore $m = 0$. So, a line with gradient zero will be perfectly flat, parallel to the x-axis.

Finally, lines can also point downwards. This means that there is negative change in the vertical direction when we make a positive change in the horizontal direction. Put differently, when we move to the right, the line moves downward. In this case, the gradient m is a negative number. This may all be summarized as follows.

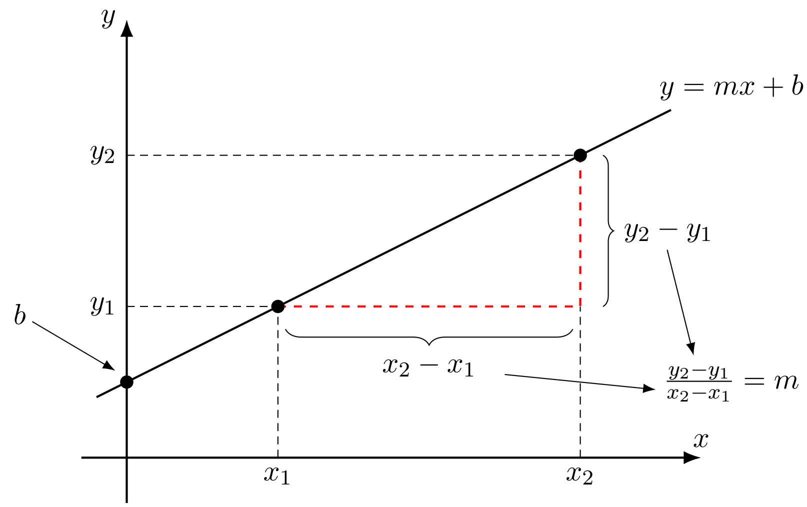


2.6 Linear Equations Revisited

Straight lines can be represented using the equation $y = mx + b$ where here:

- $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ is the gradient of the line,
- b is the y-intercept, that is, the place where the line crosses the y-axis.

This is why we called equations of the form $y = mx + b$ *linear* – the graph of their solutions is always a straight line.



This formula perfectly describes a straight line: b tells us where to start on the y-axis, and the gradient m tells us how the line moves. This is the ideal setup for a linear model: we have an equation with two fixed values m and b, and these tell us what kind of line to use in our model.

3 Worked Examples of Linear Models

We will now complete our discussion of linear models with some worked examples. There are four examples: the taxi, the heart rate monitor, a savings strategy, and finally we will solve a linear equation in order to convert Celsius into Fahrenheit, and vice-versa.

3.1 The Taxi

Suppose that you are trying to take a taxi home. The taxi driver has a rule: as an initial cost, you must pay 200 yen. Then, you must pay 200 yen per kilometer driven after that. Suppose that your home is 8 km away. How much money will you need to pay to get home?

Our first goal is to extract the relevant information, and to write a linear equation that models the overall cost. According to the rules listed above, the total cost will be given by the formula

$$\text{Total cost} = \text{Rate} \times \text{Distance} + \text{Initial Cost}$$

If we denote the Total cost by T, the Taxi's rate by R, the Distance by D and the initial cost by I, then the equation above takes the following form:

$$T = RD + I.$$

According to the statement of the problem, we are given that $I = 200$, $R = 200$, and $D = 8$, so we may calculate T directly:

$$T = (200 \cdot 8) + 200 = 1600 + 200 = 1800 \text{ yen}.$$

Suppose now that you drive 20 km, and the taxi driver has an initial cost of 200 yen. Assuming that you paid 1200 yen in total, how much was the taxi rate?

Given $T = 1200$, $D = 20$, and $I = 200$, we solve

$$1200 = 20R + 200$$

so

$$1000 = 20R \quad \Rightarrow \quad R = 50.$$

The taxi rate must have been 50 yen per km.

Finally, suppose that you drive 12 km, and the taxi rate is 300 yen per km. Assuming that you paid 2000 yen in total, what was the initial cost?

In this case, we are solving for I using $T = R \cdot D + I$:

$$2000 = 300 \cdot 12 + I = 3600 + I \quad \Rightarrow \quad I = 2000 - 3600 = -1600.$$

So the initial cost is negative! From this, we conclude that either the Taxi driver gave us 1600 yen, or more likely, in this situation you simply could not afford to take the taxi.

3.2 The Heart Rate Monitor

Suppose that you are running on a treadmill, and your heart rate is measured twice:

$$(v_1, HR_1) = (8, 125), \qquad (v_2, HR_2) = (14, 167).$$

Assume the resting heart rate is 69 bpm, and that heart rate has a linear relationship with speed. Here are our questions:

(1) Write a linear model $HR = mv + b$.
(2) Use the model to estimate HR at $v = 12$ km/h.
(3) The runner wants to keep HR at about 132 bpm. Estimate the treadmill speed.
(4) Explain why using this model to predict HR at $v = 25$ km/h might be unreliable.

Here are the solutions, in order:

(1) We can calculate the gradient m using the standard formula of Section 2. In this case, we have:

$$m = \frac{167 - 125}{14 - 8} = \frac{42}{6} = 7.$$

We are also told that the resting heart rate is 69bpm, i.e. when $v = 0$. This immediately gives us the point where our line intercepts the vertical axis (in this case, the y-axis is heart rate). Alternatively, we can select one of the data points and solve for b. For example, using $(v, HR) = (8, 125)$, we get:

$$b = HR - mv = 125 - 7 \cdot 8 = 125 - 56 = 69,$$

which is consistent. So, our linear model is:

$$HR = 7v + 69.$$

(2) Here, our task is to simply solve the linear equation assuming that $v = 12$. We substitute this into our linear model and compute the associated value of HR:

$$HR = 7 \cdot 12 + 69 = 84 + 69 = 153 \text{ bpm}.$$

(3) In this case, we are given a value of HR instead. So, we substitute this into our model and solve for v:

$$132 - 69 = 7v \Rightarrow 63 = 7v \Rightarrow v = \frac{63}{7} = 9 \text{ km/h}.$$

(4) If we try to compute the HR value associated to $v = 25$, we will find that $HR = 7(25)+69 = 244$ bpm. This is obviously impossible, so therefore we have found a limit to our model. We may conclude from this that our original assumption of linearity of heart rate with respect to speed will only be good within some range of speeds.[2]

3.3 Buying a New Dog

Suppose that you would like to buy a new dog, which costs 350000 yen. You have currently saved up 150000 yen, and based on your job you can save an extra 20000 yen per week. According to this information, how long will it take to save up for the dog?

To solve this question, we may model the savings as

$$\text{Money saved} = \text{initial savings} + (\text{weekly savings}) \times (\text{time}).$$

Since the target is 350000 yen, the initial savings are 150000 yen, and the weekly savings are 20000 yen/week, we solve

$$350000 = 150000 + 20000t.$$

Subtracting 150000 from both sides gives:

$$200000 = 20000t \quad \Rightarrow \quad t = 10.$$

Therefore it will take 10 weeks to save up for the dog.

3.4 Temperature Conversion

Your friend asks you to hang out at the park after class. That sounds nice, so you agree. He then says "make sure that you bring a coat! It will be 52 degrees outside". You think that this sounds odd, since why on Earth would you need a coat in 52 degree heat? Then you realise that your friend is American, and he is probably talking about a different temperature system: Fahrenheit.

We can convert Celsius to Fahrenheit by using the linear equation

$$F = \frac{9}{5}C + 32.$$

[2] To make this more clear: imagine that we picked a speed of 100km/h. Firstly, this is impossible because humans cannot run that fast. Secondly, if we find the associated heart rate according to our model, it will be $HR = 769$bpm, which is again impossible. However, *that is what our model predicts.* These values should not be understood as some genuine fact about human biology, but instead an obvious limitation of our model.

To see the difference between the two temperature systems, let's first set $F = 52$ and solve for C:

$$52 = \frac{9}{5}C + 32 \Rightarrow 20 = \frac{9}{5}C \Rightarrow C = 20 \cdot \frac{5}{9} = \frac{100}{9} \approx 11.1^{\circ}C.$$

Now, we may also do the opposite and set $C = 52$ and compute the associated value of F:

$$F = \frac{9}{5} \cdot 52 + 32 = \frac{468}{5} + 32 = \frac{468}{5} + \frac{160}{5} = \frac{628}{5} = 125.6^{\circ}F.$$