MAT120: Lecture 8 Handout

Non- Linear Models

In the previous lecture we introduced *linear models*, where relationships between variables are described using straight lines. In this lecture we move on to *non-linear models*. These are models where the relationship between variables is no longer a straight line. The most important example for us will be *quadratic models*, which arise naturally in geometry, physics, and many real-world situations. We will also briefly discuss inverse square laws and see several applied examples.

1 Quadratic Equations

1.1 What is a Quadratic Equation?

A quadratic equation in one variable is any equation that can be written in the form

$$ax^2 + bx + c = 0.$$

Here a, b, and c are real numbers, and $a \neq 0$.

If we took $a = 0$, then the equation would no longer be quadratic. Instead, it would reduce to

$$bx + c = 0,$$

which is a linear equation. In this sense, we can think of quadratic equations as the next least complicated type of equation after linear equations.

Exercise 1

Determine whether each of the following equations is quadratic.

(a) $2x^2 + 4x - 2 = 0$
(b) $3x - x^2 + 1 = 0$
(c) $z^2 + 3z - 2 = 0$
(d) $x^2 - 1 = 0$
(e) $x - 2 = 0$

Solution

All of these are quadratic equations *except* **(e)**. The equation $x - 2 = 0$ is linear (it has no x^2 term).

1.2 Checking Solutions

Remember: to check whether a value of x is a solution to an equation, we substitute that value into the equation and see whether the left-hand side becomes 0.

Exercise 2

Determine whether the following values of x are solutions to the quadratic equation

$$x^2 + 3x + 2 = 0.$$

(a) $x = 2$
(b) $x = 0$
(c) $x = -1$
(d) $x = -2$

Solution

(a) $x = 2$: $2^2 + 3(2) + 2 = 4 + 6 + 2 = 12 \neq 0$, so $x = 2$ is not a solution.
(b) $x = 0$: $0^2 + 3(0) + 2 = 0 + 0 + 2 = 2 \neq 0$, so $x = 0$ is not a solution.
(c) $x = -1$: $(-1)^2 + 3(-1) + 2 = 1 - 3 + 2 = 0$, so $x = -1$ is a solution.
(d) $x = -2$: $(-2)^2 + 3(-2) + 2 = 4 - 6 + 2 = 0$, so $x = -2$ is a solution.

2 Finding Solutions to Quadratic Equations

2.1 Methods for Finding Solutions

When we discussed linear equations last lecture, we saw that finding solutions to linear equations is relatively simple: we simplify the expressions in the equation, and then we rearrange everything so that the variable x is isolated on the left-hand side of the equation, and a number is on the right.

In contrast to this, solving a quadratic can often be much more difficult. In fact, sometimes it can be impossible (at least if we restrict ourselves to real number solutions). Our strategy of isolating x on one side doesn't really work well when we have an equation like $ax^2 + bx + c = 0$. There are now two terms containing x, and one of these is an x^2 term, so it can be difficult to rearrange a quadratic to get x on one side.

In this course, we will only spend one lecture discussing quadratics. Therefore, we won't be able to go into their full theory. Instead, we will focus on three main methods:

- factorization,
- using square roots,
- the quadratic formula.

Which method we use often depends on the specific equation and the context of the problem.

2.2 Factorization

Sometimes we can find solutions to a quadratic equation by rewriting it as a product of two brackets. We can imagine this as a sort of opposite of simplification: *factorization* is the process of repackaging a quadratic into brackets.

As an example, consider the expression $x^2 + 3x + 2$. We factorize the quadratic by rewriting it as a product:

$$x^2 + 3x + 2 = (x + 1)(x + 2).$$

How do we know that these two expressions are equal? We can double-check by expanding the expression on the right-hand side. The expression $(x + 1)(x + 2)$ can be expanded by applying the distributive property:

$$\begin{aligned}(x + 1)(x + 2) &= (x + 1)x + (x + 1)2 \\ &= (x^2 + x) + (2x + 2) \\ &= x^2 + 3x + 2.\end{aligned}$$

The final equality confirms that $(x + 1)(x + 2)$ is the factorization of $x^2 + 3x + 2$.

2.2.1 Using Factorization to Solve a Quadratic Equation

Suppose that we have two real numbers a and b, such that $a \times b = 0$. The only way that this product could possibly equal zero is when at least one of the factors is zero:

$$\text{if } a \times b = 0, \text{ then either } a = 0 \text{ or } b = 0 \text{ (or both).}$$

This observation is known as the *zero-factor property*.

Now consider the quadratic equation

$$x^2 + 3x + 2 = 0.$$

We can factorize the left-hand side and rewrite the equation as

$$(x + 1)(x + 2) = 0.$$

This says the product of $(x + 1)$ and $(x + 2)$ equals zero, so by the zero-factor property we conclude that either

$$x + 1 = 0 \quad \text{or} \quad x + 2 = 0.$$

Solving these gives $x = -1$ and $x = -2$. Therefore, both $x = -1$ and $x = -2$ are solutions to $x^2 + 3x + 2 = 0$ (as confirmed in Exercise 2).

2.2.2 How to Factorize (When $a = 1$)

Let's consider the easier situation in which the leading coefficient of the quadratic is $a = 1$. In this case, factorization looks like

$$x^2 + bx + c = (x + m)(x + n).$$

Expanding the right-hand side gives

$$(x + m)(x + n) = x^2 + mx + nx + mn = x^2 + (m + n)x + mn.$$

Therefore, we are looking for two numbers m and n such that

$$m + n = b \quad \text{and} \quad mn = c.$$

Finding such a pair (m, n) automatically gives the factorization.

Example: $x^2 + 3x + 2 = (x + 1)(x + 2)$ because $1 + 2 = 3$ and $1 \cdot 2 = 2$.

Exercise 3

Factorize the following quadratics.

(a) $x^2 + 10x + 9$
(b) $x^2 + 5x + 6$
(c) $x^2 + 4x + 4$

Solution

(a) We need two numbers that add to 10 and multiply to 9. The numbers 9 and 1 work, so

$$x^2 + 10x + 9 = (x + 9)(x + 1).$$

(b) We need two numbers that add to 5 and multiply to 6. The numbers 3 and 2 work, so

$$x^2 + 5x + 6 = (x + 3)(x + 2).$$

(c) We need two numbers that add to 4 and multiply to 4. The only choice is 2 and 2, so

$$x^2 + 4x + 4 = (x + 2)(x + 2) = (x + 2)^2.$$

2.3 Square Roots

Squaring a number has an inverse operation, called the square root. We write the square root as $\sqrt{\ }$. For example, if

$$x^2 = a,$$

then

$$x = \sqrt{a} \quad \text{or} \quad x = -\sqrt{a}.$$

There are two solutions because both a positive and a negative number square to the same value. For instance, the solutions to $x^2 = 25$ are $x = 5$ and $x = -5$.

Exercise 4

Compute the solutions (over the real numbers).

(a) $x^2 = 16$

(b) $x^2 = 81$

(c) $x^2 = -1$

(d) $x^2 - 1 = 0$

(e) $x^2 - 9 = 0$

(f) $x^2 + 2 = 18$

Solution

(a) $x^2 = 16$ has solutions $x = 4$ and $x = -4$.

(b) $x^2 = 81$ has solutions $x = 9$ and $x = -9$.

(c) $x^2 = -1$ has no real solution (every real number squared is non-negative).

(d) $x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

(e) $x^2 - 9 = 0 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.

(f) $x^2 + 2 = 18 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$.

2.4 The Quadratic Formula

There is a general formula that can be used to find solutions to any quadratic equation of the form

$$ax^2 + bx + c = 0.$$

The solutions are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We need $b^2 - 4ac \geq 0$ because otherwise we would be taking the square root of a negative number, which isn't possible for real numbers.

2.5 The Inverse Square Law

There is a well-known mathematical relationship called the *inverse square law*. It has the form

$$y = \frac{a}{x^2}.$$

Rearranging gives $x^2 = \frac{a}{y}$ and therefore

$$x = \sqrt{\frac{a}{y}} \quad \text{or} \quad x = -\sqrt{\frac{a}{y}}.$$

If x represents a distance, then we only use the positive value.

This type of relationship appears in many areas of science. Examples include:

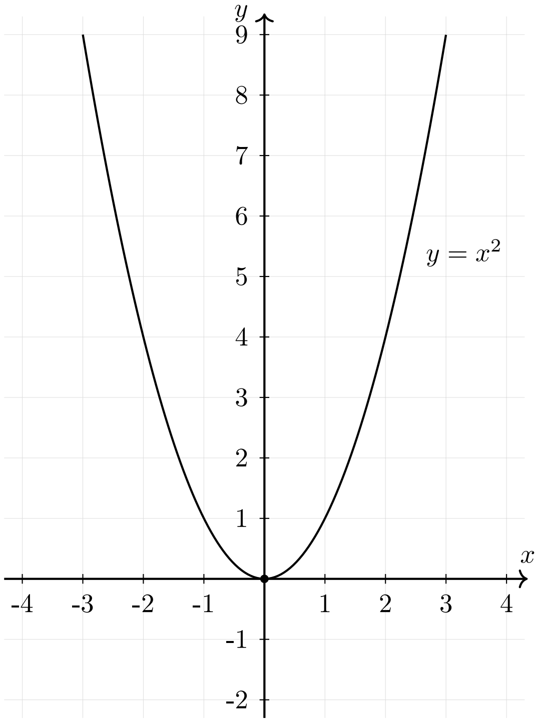
- Newton's law of gravitation,
- Coulomb's law of electric force,
- brightness of light over distance,
- temperature from a heat source,
- strength of magnetic fields.

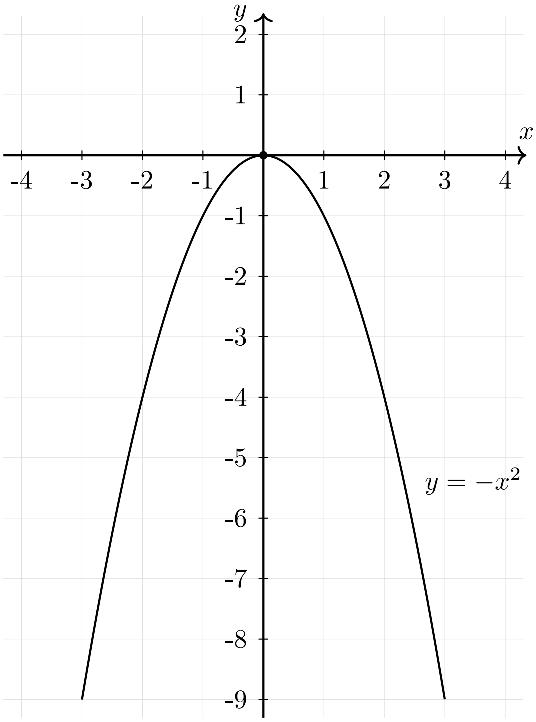
3 The Geometry of Quadratics and the Inverse Square Law

3.1 Quadratic Graphs

Quadratic equations correspond to curved graphs called *parabolas*. These curves are different from straight lines, because they do not have a constant gradient. Instead, they have a rate of change that speeds up or slows down, depending on where you are on the graph.

Generally, a quadratic equation $y = ax^2 + bx + c$ will have a graph that is shaped like a U: it turns around halfway along the graph and curves back in the other direction. Depending on the equation, the parabola will change its shape.





- The graph of $y = x^2$ turns around at the origin and points upwards. It is symmetric about the y-axis, since $(-x)^2 = x^2$.
- The graph of $y = -x^2$ also turns around at the origin, but points downwards.
- In general, if $a > 0$ then the parabola points upwards, and if $a < 0$ then the parabola points downwards. Parabolas are always symmetric about their turning points.

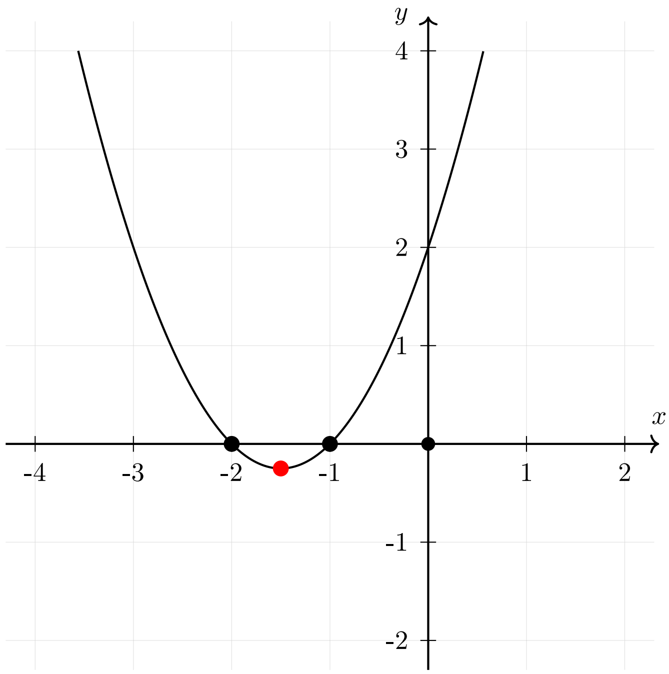
3.2 Example: The equation $y = x^2 + 3x + 2$

The graph of $y = x^2 + 3x + 2$ is the set of all points (x, y) that satisfy the equation. Since the x^2 term has a positive coefficient, this graph will be U-shaped.

Importantly, the graph will cross the x-axis at two locations. The x-axis is the horizontal line for which $y = 0$. Anywhere that the graph touches the x-axis, the value of y is 0. So these two locations are solutions to the equation

$$x^2 + 3x + 2 = 0.$$

We already found that the solutions are $x = -2$ and $x = -1$. Therefore, the graph crosses the x-axis at $x = -2$ and $x = -1$.



3.3 The Turning Point (Vertex)

A quadratic graph has a special point called the turning point, or *vertex*. This is the point where the graph changes direction, and it also describes the axis of symmetry of the graph. To find the turning point of a parabola, we can rewrite the quadratic equation in *vertex form*, which makes the turning point very easy to identify. Generally, the turning point of a quadratic $ax^2 + bx + c$ has:

$$x\text{-coordinate } = -\frac{b}{2a}, \qquad y\text{-coordinate } = c - \frac{b^2}{4a}.$$

One way to understand this (using algebra that we won't develop in this course) is the identity

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

The right-hand side is called the *vertex form* of the parabola.

Example: for $y = x^2 + 3x + 2$, we have $a = 1$, $b = 3$, $c = 2$, so the turning point occurs at

$$x = -\frac{3}{2}, \qquad y = 2 - \frac{9}{4} = -\frac{1}{4}.$$

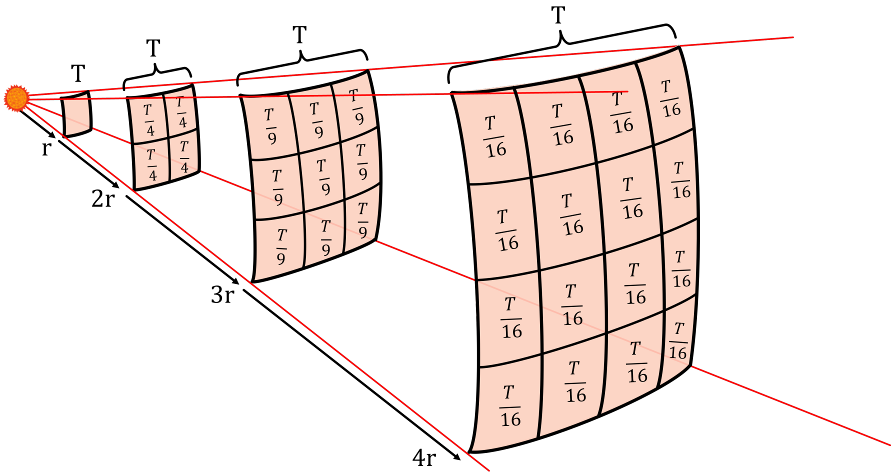
So the turning point is $\left(-\frac{3}{2}, -\frac{1}{4}\right)$, which is pictured above in red.

3.4 The Geometry of the Inverse Square Law

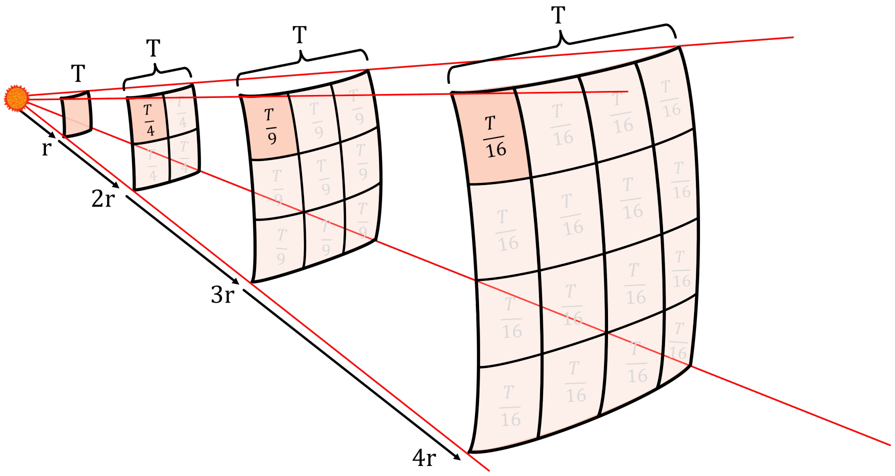
A balloon picture. Imagine a balloon, and we use a pen to draw a little square on it. If we inflate the balloon, then this square will also get bigger. If we keep inflating it more and more, then the square will get bigger and bigger. We can ask: how does the area of our little square change as the balloon gets bigger? For convenience, let's assume that the balloon is a perfect sphere with radius

r. If we double the radius of the balloon, then the area of the little square does not simply double. Instead, it becomes four times bigger. If we triple the radius, the area becomes nine times bigger. If we multiply the radius by 4, the area becomes sixteen times bigger. So, the change in area is related to r^2.

Another perspective. Consider the sun, but it is really far away, so that it is like a small dot. The sun sends out energy in all directions. Imagine a giant sphere surrounding the sun, and imagine a small square drawn on that sphere. We measure how much energy passes through this square.



Now, if we make our testing sphere twice as large, then the area of the square becomes four times larger. The total energy leaving the sun has not changed, but it is now distributed across a surface that is four times larger. This means the energy per unit area is $\frac{1}{4}$ of what it was before. If we triple the radius, the energy gets spread over a surface nine times larger, so the energy per unit area is $\frac{1}{9}$ of what it was before. This is the inverse square law.



4 Example Models and Applications

4.1 A Pirate's Cannonball

The height of an object thrown into the air can be modeled by a quadratic equation. This is known as *projectile motion*. The word *projectile* means something like a thing that is thrown through the air with force.

We can imagine firing a bullet straight up into the air. The bullet leaves the gun with a large velocity, so it climbs upward for a while. Gravity is constantly pulling the bullet down and fighting against the bullet's upward motion. Over time, gravity wins this battle: the bullet climbs less and less until it stops climbing for a moment. After this, the bullet falls back down, speeding up as it falls.

The same process holds when we fire a bullet upwards at an angle. In that case the velocity has an upward component (the *vertical velocity*) and a sideways component (the *horizontal velocity*). The vertical velocity controls how high the bullet goes.

Based on this story, we can discover a commonly used formula:

$$h(t) = h_0 + v_0 t - \frac{1}{2} g t^2.$$

Here:

- h_0 is the initial height,
- v_0 is the initial vertical velocity (how fast the object is going up),
- g is the acceleration due to gravity.

If we make some assumptions to make things easier (for example, $g = 10$ and $h_0 = 1$), this equation becomes

$$h(t) = 1 + v_0 t - 5t^2.$$

This is a quadratic with $a = -5$, $b = v_0$, and $c = 1$. It has vertex form:

$$h(t) = -5\left(t - \frac{v_0}{10}\right)^2 + \left(1 + \frac{v_0^2}{20}\right).$$

The turning point tells us the maximum height reached by the object.

Exercise 6

Suppose we have a cannon that fires a cannonball up into the air. Use the model above to determine the *maximum height* (and the time when it occurs) for:

(a) $v_0 = 20\,\mathrm{m/s}$

(b) $v_0 = 40\,\mathrm{m/s}$

Solution

The maximum occurs at the vertex, i.e. at $t = \frac{v_0}{10}$.

(a) If $v_0 = 20$, then $t = \frac{20}{10} = 2$ seconds, and

$$h(2) = 1 + 20(2) - 5(2^2) = 1 + 40 - 20 = 21.$$

So the maximum height is 21 meters, reached after 2 seconds.

(b) If $v_0 = 40$, then $t = \frac{40}{10} = 4$ seconds, and

$$h(4) = 1 + 40(4) - 5(4^2) = 1 + 160 - 80 = 81.$$

So the maximum height is 81 meters, reached after 4 seconds.

4.2 The Campfire Model

We model the temperature T at a distance d from a campfire using an inverse square law:

$$T(d) = 15 + \frac{k}{d^2},$$

where d is the distance between us and the fire, and k is a constant number that we have yet to determine.

Exercise 7

(a) Suppose the temperature at 1 meter is 55°. What is k?

(b) What is the temperature at 2 meters?

(c) What is the temperature at 0.5 meters?

(d) Suppose we want the temperature to be 45°. How far away should we sit?

Solution

(a) If $d = 1$, then $T(1) = 15 + k = 55$, so $k = 40$.

(b) At $d = 2$:

$$T(2) = 15 + \frac{40}{2^2} = 15 + \frac{40}{4} = 25^\circ.$$

(c) At $d = 0.5$:

$$T(0.5) = 15 + \frac{40}{(0.5)^2} = 15 + \frac{40}{0.25} = 175^\circ.$$

(d) If $T(d) = 45$:

$$45 = 15 + \frac{40}{d^2} \quad \Rightarrow \quad 30 = \frac{40}{d^2} \quad \Rightarrow \quad d^2 = \frac{4}{3}.$$

Therefore

$$d = \sqrt{\frac{4}{3}} \approx 1.15 \text{ meters.}$$