MAT120: Lecture 9 Handout

Exponential Models

In the previous lectures we started modelling relationships between variables. We first studied linear models (straight lines), and then we studied some basic non-linear models (especially quadratics). In this lecture we move on to *exponential models*. Exponential growth and exponential decay are famous because they appear in many real-world situations, especially when things grow (or shrink) by a *constant factor* over equal time steps.

1 The Algebra of Exponents

1.1 Multiplication and Exponentiation

Recall that multiplication is simply defined to be repeated addition. For example, multiplying a number a by an integer n means to add a to itself n times:

$$n \cdot a = a + a + \cdots + a \qquad (n \text{ times}).$$

As we saw in Lecture 5, we may extend this notion to other multiples too: the number n does not need to be an integer, and we can still define a meaningful notion of the product $n \cdot a$ that follows the usual rules of multiplication (such as commutativity and associativity).

Similarly, exponentiation by a natural number n means taking a repeated product of a with itself:

$$a^n = a \cdot a \cdot \ldots \cdot a \qquad (n \text{ times}).$$

So: multiplication is "repeated addition", and exponentiation is "repeated multiplication". When we consider an exponent like a^n, we often call a the *base* and n the *exponent*.

1.2 Rules of Exponents (for $a \neq 0$)

When the base a is the same, there are several very important rules:

(1) $a^m \cdot a^n = a^{m+n}$,

(2) $\dfrac{a^m}{a^n} = a^{m-n}$ (this requires $a \neq 0$),

(3) $(a^m)^n = a^{mn}$,

(4) $(ab)^n = a^n \cdot b^n$.

These rules allow us to manipulate exponents and turn them into simpler forms. For example:

$$2^3 \cdot 2^2 = 2^{3+2} = 2^5 = 32.$$

Why does $a^m \cdot a^n = a^{m+n}$ make sense?

Although these rules may appear random at first, they follow quite simply from the definition of exponentiation.
If we write out the left-hand side, then

$$a^m = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}}, \qquad a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}.$$

So the product $a^m \cdot a^n$ contains m copies of a and then n more copies of a, giving a total of $m+n$ copies:

$$a^m \cdot a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ times}} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{(m+n) \text{ times}} = a^{m+n}.$$

The other rules are similar, and can be justified using basic properties of multiplication such as commutativity and associativity.

1.3 Zero and Negative Exponents (for $a \neq 0$)

We can extend the idea of exponents to other numbers. When we discussed the idea of numbers in Lecture 5, we saw the *multiplicative inverse property*: any non-zero real number a has a special inverse number that multiplies with a to equal 1. We write this inverse as $\frac{1}{a}$, so that

$$a \cdot \frac{1}{a} = 1 \qquad (a \neq 0).$$

We also write this inverse as an exponent:

$$a^{-1} = \frac{1}{a} \qquad (a \neq 0).$$

This is why we require a to be non-zero: if we tried to write 0^{-1} we would be writing a fraction with zero on the bottom, which is not well-defined.

We can use exponent rules to describe other negative exponents by building off of a^{-1}. For example, since $-2 = (-1) \cdot 2$, we can use rule $(a^m)^n = a^{mn}$ to rewrite:

$$a^{-2} = a^{(-1)\cdot 2} = (a^{-1})^2 = \left(\frac{1}{a}\right)^2 = \frac{1}{a^2}.$$

Generally, for non-zero real numbers a we use the following three rules:

(1) $a^0 = 1$,

(2) $a^{-1} = \frac{1}{a}$,

(3) $a^{-m} = \frac{1}{a^m}$.

Why is $a^0 = 1$?

Pick any positive integer m. Then $0 = m - m$. Using the exponent rule $\frac{a^m}{a^n} = a^{m-n}$, we get:

$$a^0 = a^{m-m} = \frac{a^m}{a^m} = 1.$$

To see an example of rule $a^{-m} = \frac{1}{a^m}$ in action, consider a base of 2:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$$

Exercise 1

Compute the following.

(a) 3^0

(b) 2^{7-5}

(c) 4^{-2}

Solution

(a) $3^0 = 1$.

(b) $2^{7-5} = 2^2 = 4$.

(c) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$.

1.4 Rational Exponents

We can also use exponents that are not integers. The first step is to interpret exponents that are rational numbers like $\frac{1}{2}$.

Suppose $a \geq 0$. If we want $a^{1/2}$ to be consistent with exponent rules, we would expect:

$$\left(a^2\right)^{1/2} = a^{2\cdot(1/2)} = a^1 = a, \qquad \text{and} \qquad \left(a^{1/2}\right)^2 = a^{(1/2)\cdot 2} = a^1 = a.$$

So applying the exponent $\frac{1}{2}$ reverses the effect of squaring (at least for $a \geq 0$). But we already know an inverse operation for squaring: the square root. Therefore,

$$a^{1/2} = \sqrt{a} \qquad (a \geq 0).$$

More generally, for $a \geq 0$ and $n \in \mathbb{N}$ we define

$$a^{1/n} = \sqrt[n]{a}.$$

For example, since $2^3 = 8$, we have $8^{1/3} = 2$, i.e. $\sqrt[3]{8} = 2$.

1.5 Irrational Exponents

So far we have extended exponents from natural numbers to negative integers, and then to rational numbers. We will now take this a step further: it is also possible to raise a (positive) number to an exponent that is an *irrational number*.

For example: a^π, where π is the special irrational number that equals the ratio

$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

for any circle.

At first glance, it seems impossible to define a^π in terms of the ideas we already understand. But things are not as bad as they seem if we remember that irrational numbers can be approximated by rational numbers.

Recall that π has a decimal expansion

$$\pi = 3.14159\ldots$$

so if $a > 1$, we would expect a^π to be "a bit more than a^3, but less than a^4".

Approximating π by rational numbers

We can get better and better rational approximations of π by adding more digits:

Step 1: $\pi \approx 3$

Step 2: $\pi \approx 3.1 = 3 + 0.1 = 3 + \frac{1}{10}$

Step 3: $\pi \approx 3.14 = 3 + 0.1 + 0.04 = 3 + \frac{1}{10} + \frac{4}{100}$

Step 4: $\pi \approx 3.141 = 3 + 0.1 + 0.04 + 0.001 = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000}$

Step 5: $\pi \approx 3.1415 = 3 + 0.1 + 0.04 + 0.001 + 0.0005 = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000}$

and so on.

Observe that each approximation is written as a sum of rational numbers. This means we can repeatedly use the exponent rule

$$a^{m+n} = a^m \cdot a^n$$

to build approximations of a^π.

For example:

Step 1: $a^3 = a^3$
Step 2: $a^{3.1} = a^{3+\frac{1}{10}} = a^3 \cdot a^{1/10}$
Step 3: $a^{3.14} = a^{3+\frac{1}{10}+\frac{4}{100}} = a^3 \cdot a^{\frac{1}{10}} \cdot a^{\frac{4}{100}}$
Step 4: $a^{3.141} = a^{3+\frac{1}{10}+\frac{4}{100}+\frac{1}{1000}} = a^3 \cdot a^{\frac{1}{10}} \cdot a^{\frac{4}{100}} \cdot a^{\frac{1}{1000}}$

In each new step we multiply by a number of the form $a^{(\text{digit})/10^k}$, which is a power of a with a very small exponent. Since very small exponents are close to 0, the values $a^{(\text{digit})/10^k}$ are close to $a^0 = 1$. So each new step changes the answer only slightly. This is why these approximations eventually settle down (they *converge*) to a single value, which we define to be a^π.

1.5.1 Example: the irrational exponent (2^π)

To see an explicit example, let us compute 2^π. Firstly, we know that $2^3 = 8$ and $2^4 = 16$. Since π is between 3 and 4, we expect that 2^π is somewhere between 8 and 16.

Using the same step-by-step approximation idea (with a calculator), we can do:

Step 1: $2^3 = 8$
Step 2: $2^{3.1} = 2^3 \cdot 2^{0.1} \approx 8 \cdot 1.071773 \approx 8.57419$
Step 3: $2^{3.14} = 2^3 \cdot 2^{0.1} \cdot 2^{0.04} \approx 8 \cdot 1.071773 \cdot 1.028114 \approx 8.81524$
Step 4: $2^{3.141} = 2^3 \cdot 2^{0.1} \cdot 2^{0.04} \cdot 2^{0.001} \approx 8.81524 \cdot 1.000693 \approx 8.82135$
Step 5: $2^{3.1415} = 2^3 \cdot 2^{0.1} \cdot 2^{0.04} \cdot 2^{0.001} \cdot 2^{0.0005} \approx 8.82135 \cdot 1.000347 \approx 8.82441$

This procedure continues, and the answer gets closer and closer to the true value. In fact,

$$2^\pi \approx 8.82498.$$

This matches our expectation: π is between 3 and 4 (but closer to 3), so 2^π should be between 8 and 16 (but closer to 8).

1.6 Exponents with Variables

According to our discussion so far, it seems that we can meaningfully take an exponent of a (positive) number a by *any* real number b, including irrational numbers. In other words, we can always write a^b provided the base a is positive.

We can also take an exponent of a *variable*. We write a^x to mean "a raised to the power of the variable x". Since a variable is allowed to take on many values, we can imagine sliding the value of x around along the real line, which gives different values of a^x. For example:

- if $x = 2$ then $a^x = a^2$,
- if $x = 0$ then $a^x = a^0 = 1$,
- if $x = -\pi$ then $a^x = a^{-\pi}$.

In other words, the expression a^x contains information about *all* of the exponents of a, all at once.

We may also take this one step further by considering exponents that are algebraic expressions built from a single variable. For example, something like $3^{x/2}$, or 4^{x^2} is a valid exponential expression.

1.7 Solving Exponential Equations

Recall that to solve an equation in one variable means to find all values of the variable that make the equality true. The same holds for equations involving exponentials. For example, consider the equation:

$$3^x = 9.$$

This equation says: "3 to the power of something equals 9". We notice that $9 = 3^2$, so therefore solution must be $x = 2$. In contrast, $x = 3$ is *not* a solution because $3^3 = 27 \neq 9$.

Exercise 2

Solve the following equations.

(a) $2^x = 4$
(b) $2^x = 16$
(c) $2^x = 1$
(d) $2^x = \sqrt{2}$

Solution

(a) $4 = 2^2$, so $x = 2$.
(b) $16 = 2^4$, so $x = 4$.
(c) $1 = 2^0$, so $x = 0$.
(d) $\sqrt{2} = 2^{1/2}$, so $x = \dfrac{1}{2}$.

1.7.1 Solving Exponentials with Logarithms

Throughout this course, we will only solve very basic equations involving exponents (such as those in Exercise 2). However, it is worth noting that sometimes the solutions to exponential equations are not so obvious. For example, consider the equation:

$$4^x = 17.$$

We know that $4^2 = 16$ and $4^3 = 64$, so we expect that x is somewhere between 2 and 3, but perhaps very close to 2. But, based on inspection alone, this seems to be the best that we can do.

In the case of equations where the variable x is in the exponent, there is an inverse operation for exponentiation, called a *logarithm*. The logarithm that undoes 4^x is written $\log_4$, and it satisfies:

$$\log_4(4^x) = x.$$

So, to solve $4^x = 17$, we would simply apply $\log_4$ to both sides:

$$\log_4(4^x) = \log_4(17) \quad \Rightarrow \quad x = \log_4(17).$$

Using a calculator, $\log_4(17) \approx 2.04373$. More generally, for any base $n > 0$ with $n \neq 1$, there is a logarithm $\log_n$ which is the inverse of n^x.

It is worth noting that the logarithm is typically a complicated and infinitely-long decimal number that cannot be computed without a calculator. So, in this course we will not consider any equations that require logarithms to solve.

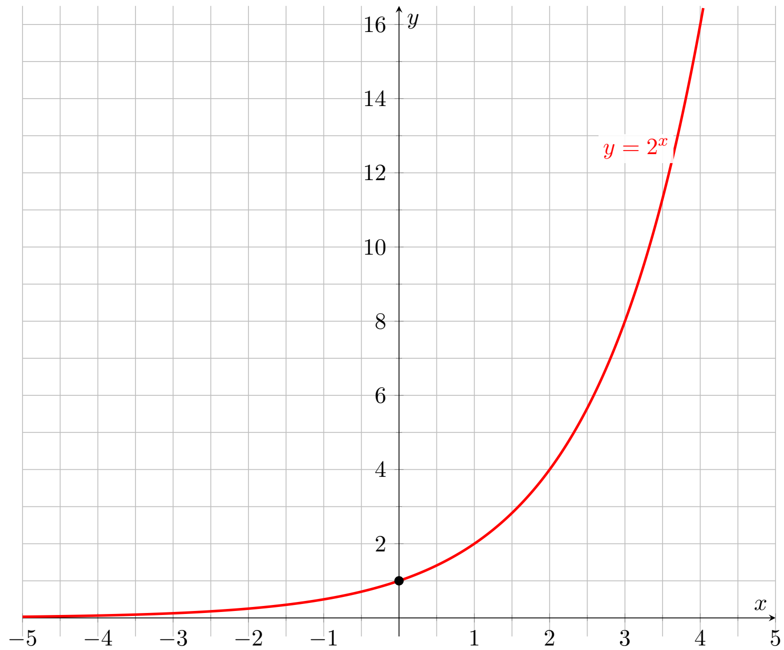
2 The Geometry of Exponentials

2.1 The Graph of $y = 2^x$

To understand the graph of $y = 2^x$, it is helpful to list some input-output values:

x	-4	-3	-2	-1	0	1	2	3	4
2^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

Based on this data, we see that the graph of $y = 2^x$ ought to pass through the y-axis at the point $(0, 1)$. We can also observe that whenever x increases by 1, the output value 2^x doubles. Similarly, whenever x decreases by 1, the output is divided by 2. In fact, we can see that 2^x grows extremely quickly as x grows. As x becomes more and more negative, 2^x gets closer and closer to 0, but it never actually reaches 0. The actual graph of $y = 2^x$ is sketched below.



2.2 What changes when we use $y = b^x$ with $b > 1$?

In the previous example, we used a value of $b = 2$ and sketched the graph of 2^x. Below you will see a sketch of exponential graphs for different values of b. There are two key facts here:

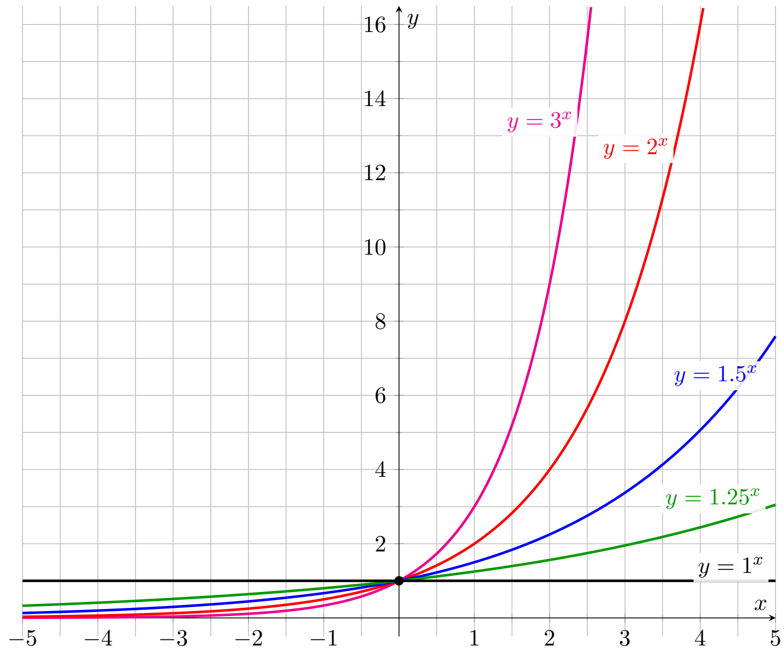
(1) For every $b > 0$, we have $b^0 = 1$, so every exponential graph passes through $(0, 1)$.

(2) The base b is called the *growth rate*, because it tells you the factor you multiply by when x increases by 1.

The larger the base b is (for $b > 1$), the faster the graph grows as x increases. For example:

- If $b = 3$ then increasing x by 1 multiplies the output by 3.
- If $b = 1.5$ then increasing x by 1 multiplies the output by 1.5.

In general, whenever we have $b > 1$, the equation $y = b^x$ is called an *exponential growth* function, and the value b is called the *growth factor*.



2.3 Exponential Decay

If $0 < b < 1$, then $y = b^x$ is an *exponential decay* function.

It still passes through $(0, 1)$, because $b^0 = 1$. But now, when x increases by 1, you multiply by a number less than 1, so the output shrinks.

Example: if $b = \frac{1}{2}$, then each step to the right multiplies the output by $\frac{1}{2}$, so the values halve each time.

As x becomes larger and larger, b^x gets closer and closer to 0. As x becomes very negative, b^x becomes very large.

2.3.1 Changing growth into decay

Using the rules of exponents:

$$b^{-x} = \frac{1}{b^x}.$$

This tells us that the graph of $y = b^{-x}$ is the same as the graph of $y = b^x$ reflected in the y-axis. But we can also rewrite

$$b^{-x} = \left(\frac{1}{b}\right)^x.$$

So reflecting the graph changes the base from b to $\frac{1}{b}$. This allows us to interpret decay as a type of "inverse" growth. For example, to model a quantity shrinking in half for every step in x, we may use the fractional growth factor:

$$2^{-x} = \left(\frac{1}{2}\right)^x.$$

2.4 A General Model of Exponential Growth

The most general exponential model we will use in this course is:

$$y = a \cdot b^{kx}.$$

Here:

- a is the initial value (starting value),
- b is the growth rate (or decay rate),
- k is a growth constant (this scales the time variable if we need to),
- x is usually time.

Again, to reiterate: for real-valued exponential models we usually assume that the growth rate b is greater than zero. There are three possibilities:

- if $0 < b < 1$ then we have **exponential decay**,
- if $b = 1$ then we have **no growth at all** (since the function is constant at $y = 1$), and
- if $b > 1$ then we have **exponential growth**.

3 Example Questions

3.1 Rice and Chess

Suppose that you put 1 grain of rice on the first square of a chessboard. On the second square, you put 2 grains of rice. On the third square, you put 4 grains of rice. On the fourth square, you put 8 grains of rice. You continue this process.

Exercise 3

A chessboard has 8×8 squares on it. How many grains of rice are on the final square?

Solution

The first square has 2^0 grains, the second has 2^1 grains, the third has 2^2 grains, and so on. Therefore the 64th square has

$$2^{63}$$

grains of rice.
Numerically,

$$2^{63} = 9{,}223{,}372{,}036{,}854{,}775{,}808 \approx 9.22 \times 10^{18}.$$

This is a number so hilariously big that it is hard to imagine. For reference: the total number of grains of rice produced on Earth in a year would be somewhere around the 46th or 47th square of the chessboard (depending on the estimate).

3.2 Bacteria growth

A scientist observes a bacterial population.

At time $t = 0$ hours, there are $N = 50$ bacteria. After 2 hours, there are $N = 500$ bacteria. After 4 hours, there are $N = 5000$ bacteria.

So the population is multiplying by a factor of 10 every 2 hours. A model for this is:

$$N(t) = 50 \cdot 10^{t/2},$$

where t is measured in hours.

Exercise 4

Answer the following.

(a) Confirm that $N(4) = 5000$.

(b) Compute $N(6)$.

(c) How many hours will it take for the population to reach 500,000 bacteria?

(d) By what factor will the population increase over 12 hours?

Solution

(a)

$$N(4) = 50 \cdot 10^{4/2} = 50 \cdot 10^2 = 50 \cdot 100 = 5000.$$

(b)

$$N(6) = 50 \cdot 10^{6/2} = 50 \cdot 10^3 = 50 \cdot 1000 = 50{,}000.$$

(c) Solve $500{,}000 = 50 \cdot 10^{t/2}$. Dividing by 50 gives $10{,}000 = 10^{t/2}$. Since $10{,}000 = 10^4$, we get $t/2 = 4$, hence $t = 8$ hours.

(d)

$$N(12) = 50 \cdot 10^{12/2} = 50 \cdot 10^6 = 50{,}000{,}000.$$

Compared to the starting value 50, this is larger by a factor of

$$\frac{50{,}000{,}000}{50} = 1{,}000{,}000.$$

3.3 The Potion Seller

Imagine that you are a great hero, and you go to a potion seller to ask for help in battle. The potion seller gives you a potion that will make you stronger. But there is a rule: the potion makes you 10 times stronger when you drink it, but you will get twice as weak every hour until you return back to your normal strength (and then we assume you stay at your normal strength after that).

A simple decay model for the *active potion effect* is:

$$S(t) = 10 \cdot \left(\frac{1}{2}\right)^t,$$

where t is measured in hours. Here $S(t)$ is your strength relative to normal *while the potion is active*. (So $S = 1$ corresponds to "normal strength"; once the model drops to 1 we interpret the potion as having worn off.)

Exercise 5

Answer the following.

(a) What will your strength be after 2 hours?

(b) At what time will you feel 1.25 times stronger than usual?

(c) How strong will you feel after 30 minutes?

Solution

(a)

$$S(2) = 10 \cdot \left(\frac{1}{2}\right)^2 = 10 \cdot \frac{1}{4} = 2.5.$$

So you feel 2.5 times stronger.

(b) Solve $1.25 = 10 \cdot \left(\frac{1}{2}\right)^t$. Dividing by 10 gives

$$0.125 = \left(\frac{1}{2}\right)^t.$$

Now $0.125 = \frac{1}{8} = \left(\frac{1}{2}\right)^3$, so $t = 3$ hours.

(c) 30 minutes is 0.5 hours. Then

$$S(0.5) = 10 \cdot \left(\frac{1}{2}\right)^{0.5} = 10 \cdot \frac{1}{\sqrt{2}} \approx \frac{10}{1.4142} \approx 7.07.$$

So after 30 minutes you feel about 7.07 times stronger.