

MAT140: Lecture 1 Handout

Arithmetic

We will begin our journey with a discussion of arithmetic. Although this may feel like a basic thing to do, later in the course we will see that many of the basic properties of numbers will ripple out into other structures. For now, you may consider this lecture as an encyclopedia of basic facts and techniques that you will certainly need to know going forward. It is most likely that these are familiar to you, however, some of the finer details may have been forgotten.

Today, we will review:

1. the number line and its properties
2. the five basic arithmetic operations: addition, subtraction, multiplication, division, and exponentiation,
3. factors, prime numbers, and prime decomposition, and finally
4. the arithmetic of fractions and decimals.

Lots of these techniques will be generalised and used later on in the course, so it is essential for you to master the basic versions now.

1 The Number Line

We will start with an almost philosophical question: what is a number? The mathematicians of old have provided several answers to this question, and we will not review them here. Instead, we will take the following approach: a number is anything that we need it to be.

Let's start small, with the idea of *quantity*. At some point in human history, we noticed that we can have more or less of objects of the same type. For example, having three coconuts is more than having two coconuts, and having two bananas is less than having four bananas. At some point we made a great leap forward in our cognition: we noticed that two coconuts and two bananas have something in common and we abstracted away from physical objects into the realm of mathematics and introduced the concept of a number. Numbers like 1,2,3 and so on are abstract representations of quantity. Later on, we eventually realised that we can also have *nothing* of something, and therefore we also introduced the number 0 to represent that. Mathematicians refer to these “counting numbers” $0, 1, 2, 3, \dots$ as *natural numbers*, and we use the funny letter $\mathbb{N}$ to denote the collection of all of them.

In an obvious sense, having 2 of something is more than having 1 of something, so there is a natural way to order all of these natural numbers:

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \dots & & & \end{array}$$

Moreover, we noticed that these quantity numbers have an operation of addition: if I have 3 bananas and I collect 2 more, then I have 5 bananas in total. After a while, we realised that alongside

Real Numbers

The set of real numbers, written $\mathbb{R}$, contains all the numbers that can be placed on the number line. Important types of real numbers include:

- **Natural numbers:** $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$.
- **Integers:** $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.
- **Rational numbers:** numbers that can be written as fractions $\frac{a}{b}$, where a, b in $\mathbb{Z}$ and $b \neq 0$.
- **Irrational numbers:** real numbers that cannot be written as fractions, such as $\sqrt{2}$, π , and e .

1.1 Absolute Values

Before getting to the “big five” operations, we will start with a brief discussion of the absolute value operation. In very simple terms, the absolute value is an operation that removes the minus sign in front of a number. We write the operation with two vertical bars, for example

$$|-3| = 3, \quad \text{and} \quad |-8| = 8.$$

In the situation that there is no minus sign to remove, then the absolute value operation does absolutely nothing, for example

$$|2| = 2, \quad \text{and} \quad |15| = 15.$$

As a matter of fact, the absolute value operation has a geometric meaning on our number line: it gives us a way to express *distances*.

Absolute Value as a Distance

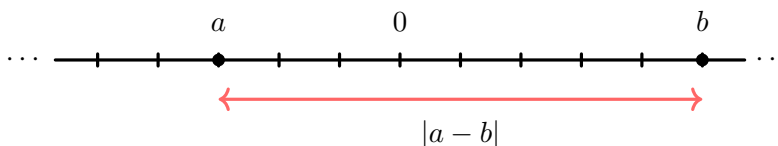
For any number a in $\mathbb{R}$, the absolute value $|a|$ gives the distance from a to 0 on the number line. Since distances are never negative, we always have

$$|a| \geq 0.$$

We may also use this idea to calculate the distances between different real numbers. The distance between two numbers a and b is given by

$$|a - b|.$$

Graphically, the operation $|a - b|$ is doing something like the following:



Exercise 1

Compute the following.

- | | |
|-----------|-------------|
| 1. $ -3 $ | 4. $- -1 $ |
| 2. $ 7 $ | 5. $ 0 $ |
| 3. $- 4 $ | 6. $- 3-5 $ |

2 Arithmetic Operations

We will now set about describing some arithmetic operations on the real numbers. In reality, there are *many* things you can do with the real number line. For now, we will introduce the five basic operations. These are:

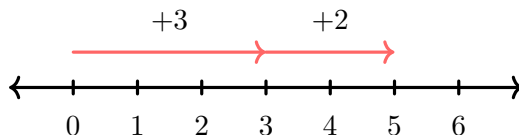
1. **Addition:** combining quantities or changes.
2. **Subtraction:** difference, or adding the opposite.
3. **Multiplication:** repeated addition or scaling.
4. **Division:** splitting into groups, or multiplying by the reciprocal.
5. **Exponentiation:** repeated multiplication.

2.1 Addition

Addition combines two numbers to make a third number. Of course, we may understand addition in terms of combinations of quantities, for example $3 + 2 = 5$ can be understood as:

$$\bullet \bullet \bullet + \bullet \bullet = \bullet \bullet \bullet \bullet \bullet$$

However, on the number line we may also interpret 3 and 2 as continuous *lengths*. That means addition would combine these lines together to create a larger line of length 5:



Using either of these interpretations, we can discover a collection of basic properties of addition that hold for all real numbers. Below are four of these important properties.

Properties of Addition

For all real numbers a, b, c in $\mathbb{R}$:

- **Commutative property:**

$$a + b = b + a.$$

For example, $3 + 5 = 5 + 3$.

- **Associative property:**

$$(a + b) + c = a + (b + c).$$

For example, $(2 + 6) + 5 = 2 + (6 + 5)$.

- **Additive identity:**

$$a + 0 = 0 + a = a.$$

- **Additive inverse:**

$$a + (-a) = 0.$$

Recall that for large nonnegative integers, we can add vertically using the carrying algorithm. For example:

$$\begin{array}{r} 1 \quad 1 \\ 1 \quad 4 \quad 8 \\ \quad 6 \quad 2 \\ + \quad 5 \quad 3 \quad 6 \\ \hline 7 \quad 4 \quad 6 \end{array} \quad \text{so} \quad 148 + 62 + 536 = 746.$$

Here we start in the right-most column and add those numbers. If the answer exceeds 10, then we add an extra 1 into the next column and repeat the procedure until we run out of columns.

2.2 Subtraction

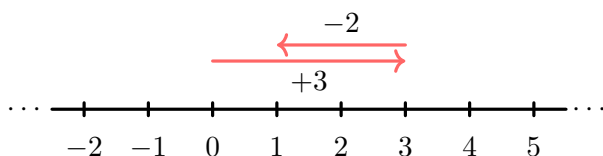
Subtraction is really addition of the opposite:

$$3 - 2 = 3 + (-2).$$

For this reason, subtraction is closely related to the additive inverse property: for any real number a , subtracting a from a quantity is the same as adding the number $-a$ to it. In this sense, subtraction and addition are opposite operations. In terms of quantities, subtraction is simply the act of *removing* things. For instance, $3 - 2 = 1$ is:

$$\bullet \bullet \bullet - \bullet \bullet = \bullet$$

If we instead choose to interpret numbers as continuous lengths, then subtraction is the removal of a piece of line, or equivalently, adding a line in the opposite direction:



For subtraction problems involving two nonnegative integers, we can use the borrowing algorithm. For example:

$$\begin{array}{r} 1 \ 3 \ 17 \\ 1 \ ~~4~~ \ ~~7~~ \\ - \quad 2 \ 9 \\ \hline 1 \ 1 \ 8 \end{array} \quad \text{so} \quad 147 - 29 = 118.$$

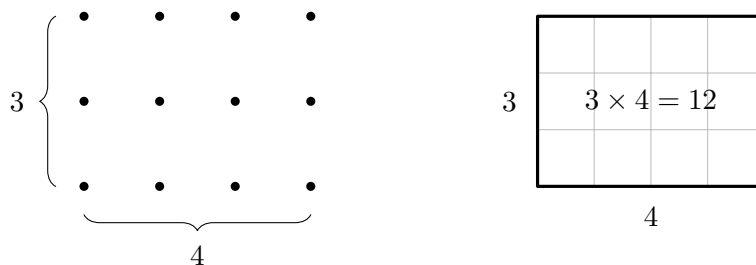
The idea here is that the 7 in the units column is too small to subtract 9, so we “borrow” 1 ten from the tens column. This turns 47 into 3 tens and 17 ones.

2.3 Multiplication

For positive integers a and b , multiplication is simply just repeated applications of addition:

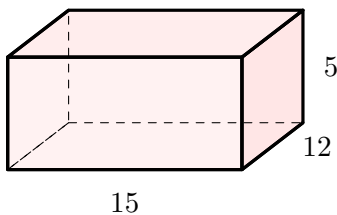
$$a \times b = \underbrace{b + b + \cdots + b}_{a\text{-many times}}.$$

We often write multiplication as either $a \times b$ or $a \cdot b$. Multiplication also has a geometric meaning: if length is represented by a line, then multiplication of two lengths gives a two-dimensional *area*.



Multiplication of three lengths gives a volume. For instance, a rectangular box with length 15, width 12, and height 5 has volume

$$15 \times 12 \times 5 = 900.$$



As with addition, multiplication has some important properties.

Properties of Multiplication

For all real numbers a, b, c in $\mathbb{R}$:

- **Commutative property:**

$$ab = ba.$$

For example, $4 \cdot (-7) = (-7) \cdot 4$.

- **Associative property:**

$$(ab)c = a(bc).$$

For example, $(3 \cdot 5) \cdot 2 = 3 \cdot (5 \cdot 2)$.

- **Multiplicative identity:**

$$a \cdot 1 = 1 \cdot a = a.$$

- **Multiplicative inverse:**

$$a \cdot a^{-1} = a^{-1} \cdot a = 1, \quad a \neq 0.$$

Equivalently,

$$a \cdot \frac{1}{a} = 1, \quad a \neq 0.$$

For large integers, we can multiply using another of these vertical algorithms. For example:

$$\begin{array}{r} 23 \\ \times 14 \\ \hline 92 \\ 230 \\ \hline 322 \end{array} \quad \text{so} \quad 23 \times 14 = 322.$$

We will practice this more in the exercise classes.

2.4 Division

In the same sense that subtraction is really just another form of addition, division is really just another form of multiplication. This time using the multiplicative inverse property above, dividing by a nonzero number is the same as multiplying by its reciprocal:

$$8 \div 2 = 8 \times 2^{-1} = 8 \times \frac{1}{2} = 4.$$

Division Vocabulary

The result of dividing one number by a nonzero number is called the **quotient**. The division

$$30 \div 6$$

may also be written as

$$30/6 \quad \text{or} \quad \frac{30}{6}.$$

In the expression $30 \div 6$, the number 30 is the **dividend** and 6 is the **divisor**. In the fraction $\frac{30}{6}$, the number 30 is the **numerator** and 6 is the **denominator**.

Some important rules for division are:

- Zero divided by a nonzero integer is 0.
- A nonzero integer divided by zero is undefined.
- The quotient of two nonzero integers with the same sign is positive.
- The quotient of two nonzero integers with different signs is negative.

For example:

$$\frac{-42}{-6} = 7, \quad 36 \div (-9) = -4, \quad 0 \div (-13) = 0, \quad -97 \div 0 \text{ is undefined.}$$

For large numbers, we can use long division. For instance:

$$\begin{array}{r} 27 \\ 13 \overline{)351} \\ \underline{26} \\ 91 \\ \underline{91} \\ 0 \end{array} \quad \text{so} \quad 351 \div 13 = 27.$$

2.5 Factors and Prime Numbers

Factors and Prime Numbers

Let a and b be positive integers. We say that a is a **factor** or **divisor** of b if a divides evenly into b .

For example, 1, 2, 3, and 6 are factors of 6.

Positive integers can be classified as follows:

- The number 1 is neither prime nor composite.
- A **prime number** is an integer greater than 1 whose only positive factors are 1 and itself.
- A **composite number** is an integer greater than 1 with more than two positive factors.

Every composite number can be written as a product of prime numbers. This is called its **prime decomposition** or **prime factorisation**. For example:

$$84 = 2 \cdot 42 = 2 \cdot 2 \cdot 21 = 2 \cdot 2 \cdot 3 \cdot 7.$$

Similarly,

$$78 = 2 \cdot 39 = 2 \cdot 3 \cdot 13.$$

The prime decomposition of an integer greater than 1 is always unique up to order, and therefore in a way it characterises the number. This is why we call prime numbers “prime” – they can be used to build all integers greater than 1 by taking products of them.

Exercise 2

Write the prime factorisation of each number.

1. 68

2. 133

3. 43

2.6 The Distributive Property

The distributive property explains how multiplication interacts with addition:

$$a(b + c) = ab + ac.$$

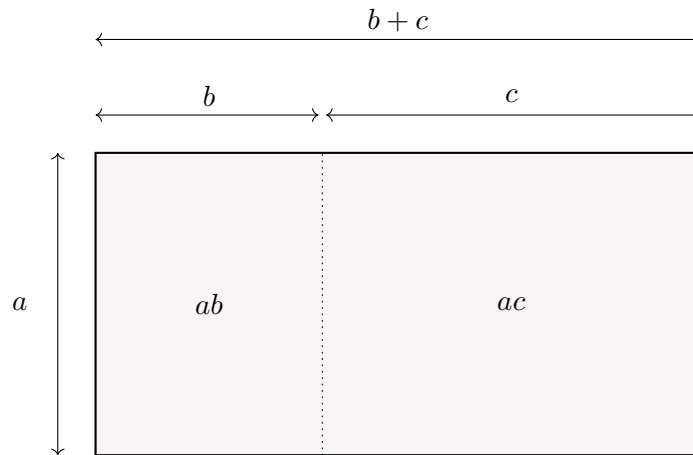
Equally,

$$(a + b)c = ac + bc.$$

For example:

$$3(8 + 5) = 3 \cdot 8 + 3 \cdot 5.$$

The distributive property has a geometric interpretation. The area of one large rectangle can be split into the areas of two smaller rectangles.



The total area can be computed in two equivalent ways:

$$\text{Total area} = a(b + c)$$

or

$$\text{Total area} = ab + ac.$$

Therefore:

$$a(b + c) = ab + ac.$$

2.7 Exponentiation

Multiplication is repeated addition. For a positive integer b , exponentiation is repeated multiplication:

$$a^b = \underbrace{a \cdot a \cdot \dots \cdot a}_{b\text{-many times}}.$$

Here a is called the **base**, and b is called the **exponent**.

For example:

$$2^3 = 2 \cdot 2 \cdot 2 = 8.$$

Rules of Exponents

For nonzero real numbers a and b , and integers m and n , we have:

$$\begin{aligned}a^m a^n &= a^{m+n}, \\ \frac{a^m}{a^n} &= a^{m-n}, \\ (a^m)^n &= a^{mn}, \\ (ab)^n &= a^n b^n.\end{aligned}$$

For example:

$$2^3 \cdot 2^2 = 2^{3+2} = 2^5 = 32.$$

We will discuss these rules at length in the later lectures. For now, it is enough to simply remember that they exist.

2.8 Order of Operations

When an expression contains several operations, we need a standard order for evaluating it.

Order of Operations

The usual order of operations is:

1. **P**arentheses first,
2. then **E**xponents,
3. then **M**ultiplication and **D**ivision,
4. and finally **A**ddition and **S**ubtraction.

The abbreviation PEMDAS is commonly used.

As an example, suppose we have some real number a , and write something like:

$$3(a - 7)^2 + 2.$$

According to our order of operations, this means:

subtract 7, then square the result, then multiply by 3, then add 2.

You should be careful when manipulating negative numbers and brackets. For instance, observe that:

$$(-3)^2 = (-3)(-3) = 9,$$

but

$$-3^2 = -(3^2) = -9,$$

since in the latter, the exponent comes *first* before the minus sign.

Exercise 3

Evaluate the following.

1. -3^2

2. $(-3)^2$

3. $5 + 2^2$

4. $(5 + 2)^2$

5. $3 \cdot 2^2$

6. $(3 \cdot 2)^2$

3 Fractions

A fraction

$$\frac{a}{b}$$

means $a \div b$, where $b \neq 0$. Fractions can represent:

- a part of a whole,
- a ratio,
- a rate, such as “per”.

For example:

$$\frac{3}{4} = 0.75.$$

3.1 Simplifying Fractions

To simplify a fraction, divide the numerator and denominator by the same nonzero number. The goal is to write the fraction in simplest form, meaning that the numerator and denominator have no common factor besides 1.

For example:

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}.$$

One method is to use prime decomposition:

$$\frac{25}{100} = \frac{5 \cdot 5}{2 \cdot 2 \cdot 5 \cdot 5} = \frac{1}{2 \cdot 2} = \frac{1}{4}.$$

Another method is to use the greatest common factor. The greatest common factor of two positive integers is the largest number that divides both of them. For example, the greatest common factor of 14 and 21 is 7, so

$$\frac{14}{21} = \frac{2 \cdot 7}{3 \cdot 7} = \frac{2}{3}.$$

3.2 Adding and Subtracting Fractions

To add or subtract fractions, we need a common denominator which is nonzero:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

If the denominators are different, rewrite the fractions using a common denominator. For example:

$$\frac{1}{8} + \frac{1}{4} = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}.$$

More generally, where $c, d \neq 0$:

$$\frac{a}{c} + \frac{b}{d} = \frac{ad}{cd} + \frac{bc}{cd} = \frac{ad + bc}{cd}.$$

3.3 Multiplying Fractions

To multiply fractions, multiply the numerators and multiply the denominators where the denominators are nonzero:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

For example:

$$\frac{3}{5} \cdot \frac{10}{7} = \frac{30}{35} = \frac{6}{7}.$$

3.4 Dividing Fractions

Dividing by a nonzero number is multiplying by its reciprocal:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, \quad b, c, d \neq 0.$$

For example:

$$\frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \cdot \frac{6}{5} = \frac{12}{15} = \frac{4}{5}.$$

Exercise 4

Compute the following.

1. $\frac{5}{2} + \frac{3}{4}$

2. $\frac{5}{2} \cdot \frac{3}{4}$

3. $\frac{6}{3} \div \frac{1}{2}$

4. $\frac{18}{24}$

5. $\frac{7}{8} - \frac{1}{4}$

6. $\frac{3}{5} \cdot \frac{15}{9}$

4 Decimals

Digits to the right of the decimal point represent tenths, hundredths, thousandths, and so on. For example:

$$0.4 = \frac{4}{10}, \quad 0.04 = \frac{4}{100}.$$

Also,

$$2.305 = 2 + \frac{3}{10} + \frac{0}{100} + \frac{5}{1000}.$$

4.1 Converting Between Fractions and Decimals

If the denominator is 10, 100, 1000, ..., then converting to a decimal is immediate:

$$\frac{7}{100} = 0.07.$$

Otherwise, division converts a fraction to a decimal. For example:

$$\frac{3}{4} = 0.75.$$

4.2 Rounding Decimals

Rounding Decimals

To round a decimal:

- If the next digit is less than 5, round down.
- If the next digit is 5 or greater, round up.

The phrase “round to n decimal places” means that the answer should have n digits after the decimal point.

For example, consider 4.85731.

- Rounded to 2 decimal places:

$$4.85731 \approx 4.86.$$

- Rounded to 3 decimal places:

$$4.85731 \approx 4.857.$$

- Rounded to 4 decimal places:

$$4.85731 \approx 4.8573.$$

Decimals can be added, subtracted, multiplied, and divided using the same vertical algorithms as before. The main extra detail is to keep careful track of the decimal point.

For example:

$$0.58 + 1.06 = 1.64.$$

Exercise 5

Compute the following.

1. $0.58 + 1.06$
2. Convert $\frac{7}{100}$ to a decimal.
3. Convert $\frac{3}{4}$ to a decimal.

4. Round 4.85731 to 2 decimal places.
5. Round 4.85731 to 3 decimal places.
6. Round 4.85731 to 4 decimal places.

Solutions to the Exercises

Exercise 1

1. $|-3| = 3$.
2. $|7| = 7$.
3. $-|4| = -4$.
4. $-|-1| = -1$.
5. $|0| = 0$.
6. $-|3 - 5| = -|-2| = -2$.

Exercise 2

1. $68 = 2 \cdot 34 = 2 \cdot 2 \cdot 17$.
2. $133 = 7 \cdot 19$.
3. 43 is prime, so its prime factorisation is just 43.

Exercise 3

1. $-3^2 = -(3^2) = -9$.
2. $(-3)^2 = (-3)(-3) = 9$.
3. $5 + 2^2 = 5 + 4 = 9$.
4. $(5 + 2)^2 = 7^2 = 49$.
5. $3 \cdot 2^2 = 3 \cdot 4 = 12$.
6. $(3 \cdot 2)^2 = 6^2 = 36$.

Exercise 4

1.
$$\frac{5}{2} + \frac{3}{4} = \frac{10}{4} + \frac{3}{4} = \frac{13}{4}.$$
2.
$$\frac{5}{2} \cdot \frac{3}{4} = \frac{15}{8}.$$
3.
$$\frac{6}{3} \div \frac{1}{2} = \frac{6}{3} \cdot \frac{2}{1} = 2 \cdot 2 = 4.$$
4.
$$\frac{18}{24} = \frac{3}{4}.$$
5.
$$\frac{7}{8} - \frac{1}{4} = \frac{7}{8} - \frac{2}{8} = \frac{5}{8}.$$
6.
$$\frac{3}{5} \cdot \frac{15}{9} = \frac{45}{45} = 1.$$

Exercise 5

1. $0.58 + 1.06 = 1.64$.
2. $\frac{7}{100} = 0.07$.
3. $\frac{3}{4} = 0.75$.
4. $4.85731 \approx 4.86$.
5. $4.85731 \approx 4.857$.
6. $4.85731 \approx 4.8573$.