MAT140 — Lecture 10 Handout
More on Factorization

Last lecture we introduced **factorization** as the reverse of expanding brackets. For example,

$$(x^2-1)(x-2) \longrightarrow x^3-2x^2-x+2, \qquad x^3-2x^2-x+2 \longrightarrow (x^2-1)(x-2).$$

In this lecture we learn some **special factorization patterns**, and then we see what factorization is **good for**: it lets us solve many non-linear equations. This latter feature has a deep and interesting algebraic story, which we will return to later in the course. For now, we will focus on the basics by giving a geometric description of polynomial equations.

Today we will:

1. Learn some special factorization forms: the difference of two squares, the perfect square trinomials, and the sum/difference of cubes.
2. Practice complete factorization.
3. Learn how to solve polynomial equations by using the Zero-Factor Property.
4. Study the geometric relationship between factorization and polynomial solutions.

1 Difference of Two Squares

One of the easiest special polynomial forms to recognize is

$$a^2-b^2,$$

where here a and b are some algebraic expressions. The above form is often called a *difference of two squares*. The factorization scheme is as follows.

Difference of Two Squares

If a and b are real numbers, variables, or algebraic expressions, then

$$a^2-b^2=(a+b)(a-b).$$

This general form can be verified by trying to expand the factorization using FOIL. When we do that, the cross-terms will cancel each other out, leaving us with the difference of two squares:

$$(a+b)(a-b)=a^2-ab+ab-b^2=a^2-b^2.$$

The key to working with this factorization scheme is to make the values of a and b explicit. That way, you can simply follow the above equation as a kind of formula. For example, here are three differences of two squares, and their associated values of a and b: For instance:

$$x^2-1 \quad \text{has } a=x \text{ and } b=1, \text{ so} \quad x^2-1=(x+1)(x-1)$$

But, the same scheme works when a and b are *expressions*. For example:

$$4x^2-9 \quad \text{has } a=2x \text{ and } b=3, \text{ so} \quad 4x^2-9=(2x+3)(2x-3).$$

This scheme also works when we have expressions of more than one variable. For example:

$$9x^4-16y^2\text{has } a=3x^2 \text{ and } b=4y, \text{ so} \quad 9x^4-16y^2=(3x^2+4y)(3x^2-4y).$$

Exercise 1

Factorize each polynomial.

1. $x^2 - 36$
2. $x^2 - \dfrac{4}{25}$
3. $81x^2 - 49$
4. $9a^2 - 16b^2$
5. $25y^4 - 1$

2 Complete Factorization

2.1 Prime Number Decomposition

Consider an integer, say 72. In a way, we can "factorize" this number by expressing it as a product of factors. For instance, $12 \cdot 6$ is a factorization of 72, and so is $2 \cdot 36$. Similarly, it is often the case that we can factorize the factors of 72, like $36 = 6 \cdot 6$. In principle, we can keep going with this procedure and break each factor of 72 into smaller and smaller pieces until we cannot proceed any further. If we were to do this, we would end up with the factorization:

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2.$$

As you can see, we have now represented 72 purely in terms of prime numbers which cannot be broken down further. This is called the *prime number decomposition* of 72, because we are "breaking down" the number 72 into a product of primes. It is unique, and always exists for any positive integer.

Notice also that the same prime number can appear multiple times in the same decomposition, for example 2 appears 3 times in the prime number decomposition of 72. This is called the *multiplicity* of the factor, and has the following definition.

> Multiplicity
>
> Let n be a positive integer, with prime decomposition:
>
> $$n = (p_1)^{a_1} \cdot (p_2)^{a_2} \cdot \ldots \cdot (p_m)^{a_m},$$
>
> where here p_i are prime numbers. The **multiplicity** of the prime number p_i in n is the exponent a_i.

For example, in the decomposition $72 = 2^3 \cdot 3^2$, the prime number 2 has multiplicity 3 and the prime number 3 has multiplicity 2.

2.2 Irreducible Polynomials

Polynomial factorization is similar to the prime number decomposition of integers. In some cases, we can factorize multiple times in a row to break a polynomial down into smaller and smaller pieces. For example, the factorization of the polynomial $x^4 - x^2$ can be performed in two steps:

$$x^4 - x^2 = x^2(x^2 - 1) = x^2(x + 1)(x - 1).$$

As you can see, we have to stop this process after two steps, because there is simply nothing more to do. This observation can be made clearer: notice that some polynomials can be broken down in a product of two smaller polynomials, and some cannot. The latter are much like the prime numbers: the only factors of a "prime polynomial" (informally) are the polynomial itself, and the nonzero constants. Actually, these "prime polynomials" have a technical name: they are called *irreducible polynomials.*

> Irreducible Polynomials
>
> A polynomial is called **irreducible** if it cannot be decomposed as a product of two non-constant polynomials.

For example, the polynomial $x - 1$ is irreducible, because there are no non-constant polynomials of smaller degree that multiply to give $x - 1$.

2.3 Factorizing Completely

Factorization is sometimes a multi-step process. After you factor once, you should always check whether the factors you obtained can be factorized further. This is very similar in spirit to the prime number decomposition mentioned previously. However, in this case, we are looking to keep factorizing a polynomial until we end up representing it as a product of irreducible polynomials. (Uniqueness note: over the real numbers, a complete factorization is unique up to reordering factors and pulling out a nonzero constant. In other words, you and a friend might write the factors in a different order, or distribute constants differently, but the irreducible building blocks are the same.)

> Factoring completely
>
> To factor a polynomial **completely**:
>
> 1. First factor out the **greatest common monomial factor** (if there is one).
> 2. Then factor the remaining polynomial.
> 3. Repeat until none of the factors can be reduced further using our tools.

As an example, let's consider the polynomial $x^4 - 1$. Firstly, observe that this is a difference of two squares with $a = x^2$ and $b = 1$. Therefore, we can factorize it using the scheme detailed in Section 1:

Example 1

$$x^4 - 1 = (x^2)^2 - 1^2 = (x^2 + 1)(x^2 - 1).$$

However, we can now observe that the smaller polynomial $x^2 - 1$ is *again* a difference of two squares, this time with $a = x$ and $b = 1$. Therefore, we can keep going with our factorization:

$$x^4 - 1 = (x^2)^2 - 1^2 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1).$$

Here x^2+1 cannot be factorized further, so our process ends and we have managed to represent x^4-1 as a product of irreducible polynomials.

Consider now the polynomial $20x^3-5x$. The two terms have a greatest monomial factor equal to $5x$, so we can factorize this out first, and then follow through with the rest of the factorization:

$$20x^3-5x=5x(4x^2-1)=5x\big((2x)^2-1^2\big)=5x(2x+1)(2x-1).$$

Exercise 2

Factor each polynomial completely.

1. x^4-1
2. x^4-16
3. $20x^3-5x$
4. $18y^2-2$
5. $8x^3+27$

3 Perfect Square Trinomials

We now move on to our next "special form". This is the *perfect square trinomial*, which is usually the result of squaring a binomial.

Perfect square trinomials

If a and b are numbers, variables or algebraic expressions, then:

$$a^2+2ab+b^2=(a+b)^2 \qquad \text{and} \qquad a^2-2ab+b^2=(a-b)^2.$$

Again, these rules can be easily verified by working backwards and expanding out the factorized forms. For the first case, we have:

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ab+b^2=a^2+2ab+b^2,$$

and for the second case, we have:

$$(a-b)^2=(a-b)(a-b)=a^2-ab-ab+b^2.$$

As you can see, the perfect square trinomial forms are very similar to the difference of two squares. However, in contrast to Section 1, we see that here the cross-terms don't appear twice with opposite sign and cancel – instead they appear twice with *the same* sign and *double up*.

An example of a perfect square trinomial would be something like: x^2+4x+4, which has $a=x$ and $b=2$. Therefore:

$$x^2+4x+4=(x+2)^2.$$

As an example of the case with negative sign, consider the polynomial y^2-6y+9. Again, this is a perfect square trinomial, however this time $a=y$ and $b=-3$. The factorization is:

$$y^2-6y+9=(y-3)^2.$$

It should be noted that the coefficients of the polynomial don't need to be integers. For example:

$$x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2.$$

Exercise 3

Which of the following are perfect square trinomials? Factorize the ones that are.

1. $m^2 - 4m + 4$
2. $4x^2 - 2x + 1$
3. $y^2 + 6y - 9$
4. $x^2 + x + \frac{1}{4}$
5. $9p^2 + 12pq + 4q^2$

4 Sum or Difference of Two Cubes

Our final "special form" that we will cover today is that of the sum or difference of two cubes.

Sum and difference of cubes

If a and b are numbers, variables or expressions, then:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2), \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

Again, these can both be verified by expanding out the brackets on the right hand side of each equation:

$$(a+b)(a^2 - ab + b^2) = a^3 + ba^2 - a^2b - ab^2 + ab^2 + b^3 = a^3 + b^3,$$

and:

$$(a-b)(a^2 + ab + b^2) = a^3 - ba^2 + a^2b - ab^2 + ab^2 - b^3 = a^3 - b^3.$$

Here are some examples of the sums or differences of two cubes:

$$y^3 + 27 = y^3 + 3^3 = (y+3)(y^2 - 3y + 9),$$

$$64 - x^3 = 4^3 - x^3 = (4 - x)(16 + 4x + x^2),$$

$$2x^3 - 16 = 2(x^3 - 8) = 2(x-2)(x^2 + 2x + 4).$$

It should be noted that of course, there are also general schemes for the factorization of other more complicated polynomials involving higher degrees. For instance – $a^4 - b^4$ probably has some scheme. However, practically speaking, it is not reasonable to simply memorise all of the possible ways that polynomials can be. In practice it is a better idea to learn *techniques* of factorization to help with more complicated polynomials.

Exercise 4

Factorize each polynomial.

1. $y^3 + 27$
2. $64 - x^3$
3. $2x^3 - 16$
4. $a^3 - 8$
5. $27m^3 + 1$

5 Solving Polynomial Equations by using Factorization

Up until now, we have only discussed polynomials on the level of manipulation rules. We will now take a step into geometry and study what the graphs of polynomials look like. We will see a clear payoff: we can often solve equations of the form $P(x) = 0$ by using factorization, and this has a clear geometric interpretation.

5.1 Roots and the geometry of solutions

We will start with a discussion of the solutions to polynomial equations. As a matter of fact, these have a fancy name – they are called *roots* of a polynomial.

Roots of a Polynomial

Let $P(x)$ be a polynomial. A number r is called a **root** of P if

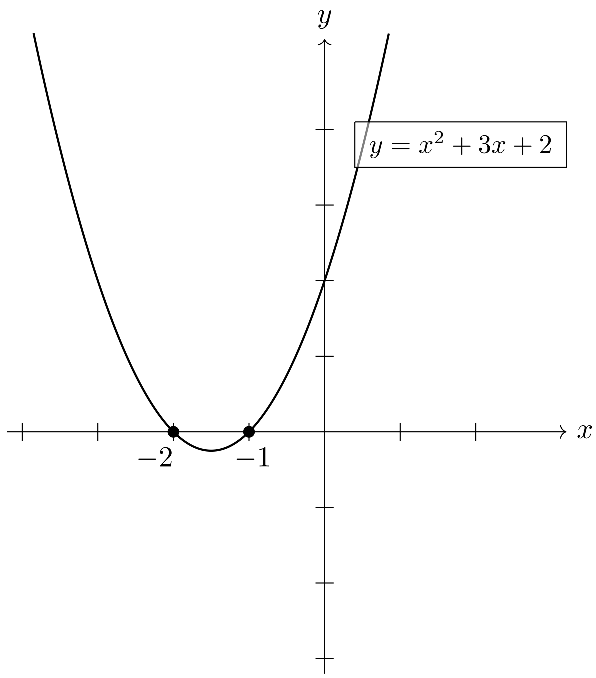
$$P(r) = 0.$$

That is, r is a root of the polynomial $P(x)$ if r is a solution to the equation $P(x) = 0$.

Let's consider a quadratic equation: $x^2 + 3x + 2$. We can observe that this equation has two roots: $x = -1$ and $x = -2$, because:

$$(-1)^2 + 3(-1) + 2 = 1 - 3 + 2 = 0 \qquad \text{and} \qquad (-2)^2 + 3(-2) + 2 = 4 - 6 + 2 = 0.$$

If we now plot the graph of the function $y = x^2 + 3x + 2$, we will get the following curve:



As you can see, the graph $y = x^2 + 3x + 2$ has two x-intercepts, namely at $x = -2$ and $x = -1$. This is no great mystery, and in fact, it actually makes *perfect sense*: the equation of the x-axis is $y = 0$, so, the equation $x^2 + 3x + 2 = 0$ can be solved by looking for any points that lie on the graph of $y = x^2 + 3x + 2$ and the graph of $y = 0$ at the same time.

Using our simple example, we have just discovered hints of a general rule.

Geometric Interpretation of Roots

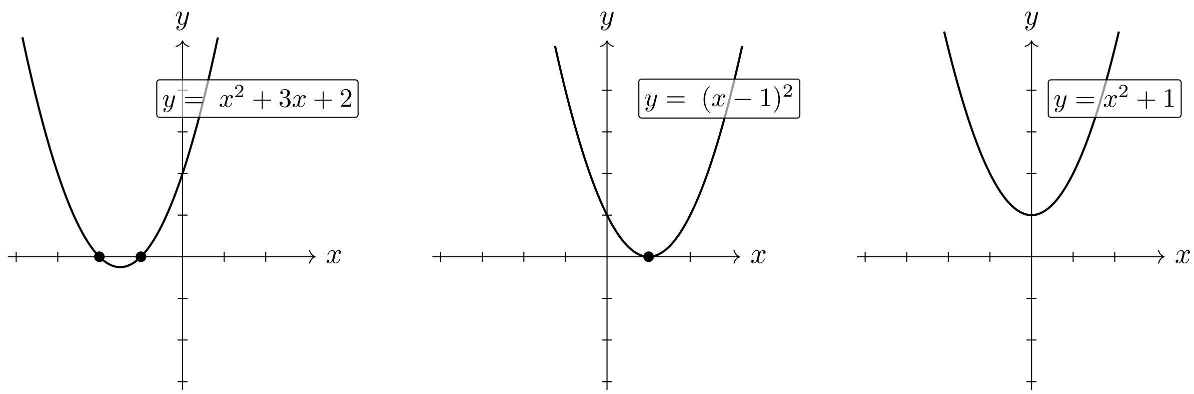
Let $P(x)$ be a polynomial. Then

1. if r is a root of $P(x)$, then the coordinates $(r, 0)$ describe a point where $P(x)$ intersects the x-axis, and conversely
2. the x-coordinate of any point at which the graph of $P(x)$ intersects the x axis describes a root of $P(x)$.

It should be noted that the graph of a polynomial can intersect the x-axis in two different ways:

- the graph can hit the x-axis and pass through the other side, or
- the graph can merely touch the x-axis and turn around.

As well as this, it is entirely possible for the graph of the polynomial to *never touch the x-axis at all.* Examples of these three situations are drawn below.



Two solutions **One solution** **No solutions**

The three pictures show three different situations:

- **Two solutions:** $y = x^2 + 3x + 2$ crosses the x-axis twice, at $x = -2$ and $x = -1$, so $x^2 + 3x + 2 = 0$ has solutions $x = -2$ and $x = -1$.
- **One real solution:** $y = (x - 1)^2$ touches the x-axis at $x = 1$ but does not cross it, so $(x - 1)^2 = 0$ has a single solution $x = 1$.
- **No real solutions:** $y = x^2 + 1$ never crosses the x-axis, so $x^2 + 1 = 0$ has no real solutions.

5.2 Solving polynomial equations using factorization

Consider again the polynomial $x^2 + 3x + 2$. This can easily be factorized:

$$x^2 + 3x + 2 = (x + 1)(x + 2).$$

Observe that the two numbers in the brackets look very similar to the two roots $x = -1$ and $x = -2$. In fact, if we were to substitute either of these values into the factorized form, we will see some nice simplifications:

- if we substitute $x = -1$ into the factorized form we get $(-1 + 1)(-1 + 2) = (0)(1) = 0$, and
- if we substitute $x = -2$ into the factorized form we get $(-2 + 1)(-2 + 2) = (-1)(0) = 0$.

So, we see that substituting in either of the solutions forces one of the products to equal zero, which guarantees that the entire expression evaluates to zero.

This observation gives us hints at a general strategy to solve polynomials: if we can factorize them then perhaps we can "read off" the roots based on the information in the brackets. The following observation is at the foundation of this approach.

Zero-Factor Property

If a and b are real numbers, variables or algebraic expressions and

$$ab = 0,$$

then it must be the case that either $a = 0$ or $b = 0$ (or both). This also works for three or more factors. If $abc = 0$, then at least one of a, b, c has to be zero.

This property is the key to solving polynomials via factorization: if we have an equation of the form $P(x) = 0$ where P is a polynomial, then we can find solutions to the equation by factorizing $P(x)$ and setting each factor equal to zero one-by-one. The general method is as follows.

Solving polynomial equations by factorization

To solve a polynomial equation by factorization:

1. Rearrange the equation so that one side is 0.
2. Factorize the polynomial on the other side completely.
3. Set each factor equal to 0.
4. Solve each simpler equation.
5. Check solutions by substitution. If you draw the graph, the solutions should match the x-intercepts.

Going back to our polynomial x^2+3x+2, we can derive the solutions to the equation $x^2+3x+2=0$ by factorizing. We have $(x+1)(x+2)=0$, so by the zero-factor property it *must* be the case that either one of these products is itself zero. There are two cases to consider.

- Case 1: $x+1=0$, which implies that $x=-1$.
- Case 2: $x+2=0$, which implies that $x=-2$.

These are the two solutions from before.

As another example, consider the polynomial $x^2-x-6=0$. To solve this equation we factorize and set the products equal to zero one-by-one. The factorization gives $(x+2)(x-3)=0$. There are again two cases to consider.

- Case 1: $x+2=0$ which implies that $x=-2$.
- Case 2: $x-3=0$ which implies that $x=3$.

As the following exercise demonstrates, our approach only works when the right-hand-side of the equation is zero.

5.2.1 Exercise 5

Solve the following polynomial equations by factorization.

1. $x^2-2x+16=6x$
2. $3x^3=15x^2+18x$

Hint: rearrange the equations so that the right-hand-side equals zero.

Exercise 6

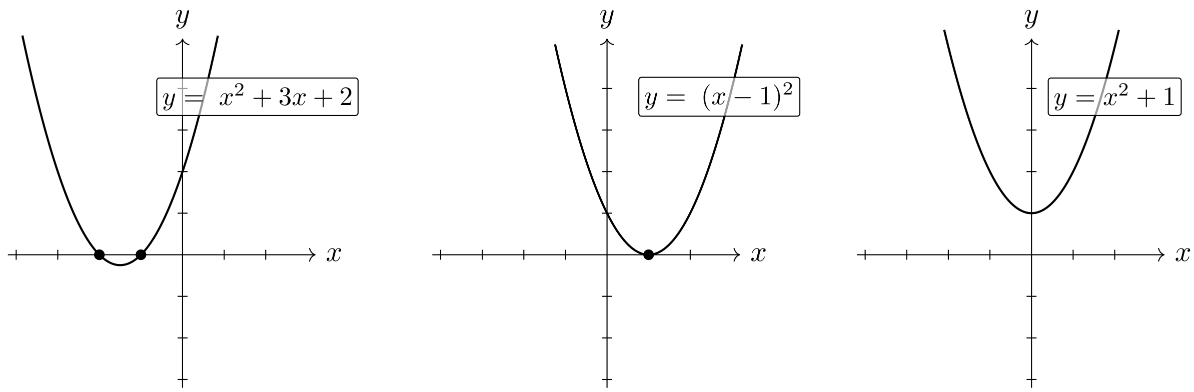
Solve each equation.

1. $x^2-4x+4=0$
2. $x^2+3x+17=1-7x$

3. $x^3 - 7x^2 + 10x = 0$
4. $5x^3 + 33x^2 + 90x = x^3 - 3x^2 + 10x$
5. $x^2 + 1 = 0$

5.3 The relationship between roots and irreducibility

Let's return to the three quadratic equations from before.



Two solutions **One solution** **No solutions**

In the first case, we have the graph of the polynomial $x^2 + 3x + 2$, which we know can be factorized as $(x + 1)(x + 2)$. In the second case, we have a graph of a polynomial that is already presented in factorized form: $(x - 1)^2$, and in the third case, we have the graph of the polynomial $x^2 + 1$, which cannot be factorized at all. This begs the question: what is the exact relationship between roots of a polynomial and its *complete* factorization?

We can explain this by digging into some algebra. Consider the polynomial $P(x) = x^3 - 6x^2 + 11x - 6$. This polynomial happens to have a root $x = 1$, since $(1)^3 - 6(1^2) + 11(1) - 6 = 1 - 6 + 11 - 6 = 0$. Let's consider now the polynomial $x - 1$. We can divide $P(x)$ by $(x - 1)$, and this will generally give some other (smaller) polynomials:

$$P(x) = (x - 1)Q(x) + R(x),$$

where here Q is the quotient and $R(x)$ is the remainder. As we saw in Lecture 8, the remainder term R always has degree smaller than the divisor. In this case, the divisor is $(x - 1)$, which is a degree-1 polynomial. This means that R must have degree 0, i.e. it is a constant term. Now, when we substitute in our root, we see that:

$$0 = P(1) = (1 - 1)Q(1) + R = 0 + R$$

which means that $R = 0$, i.e. $(x - 1)$ divides the polynomial $P(x)$ evenly. This is an instance of the following general rule.

Roots and linear factors

If r is a root of a polynomial $P(x)$, then $(x - r)$ is a factor of $P(x)$.

This means that solving $P(x) = 0$ is essentially the same as finding all of the *linear factors* of $P(x)$. Put differently: every root gives a linear factor in the complete factorization of $P(x)$. This explains why the graph of $x^2 + 1$ does not cross the x-axis: if it *did* cross the x-axis, then those coordinates could be used to factorize $x^2+1 = (x-r)(x-s)$. But, the polynomial x^2+1 is *irreducible over the real numbers* – it cannot be broken down into a product of polynomials of a smaller degree. Therefore, no such factors exist, and in turn, the graph of x^2+1 cannot possibly cross the x-axis.

Multiplicity of a root

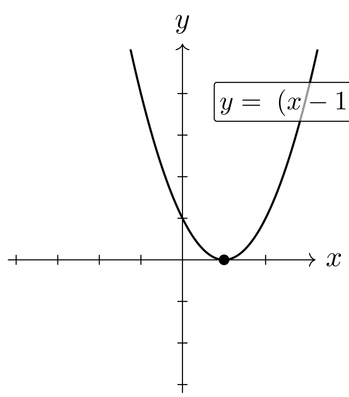
We say that r is a root of **multiplicity** k if

$$P(x) = (x-r)^k\, Q(x) \quad \text{and} \quad Q(r) \neq 0.$$

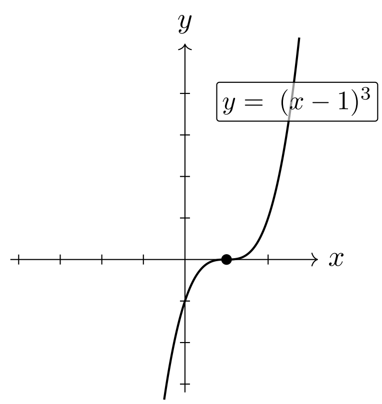
In other words: a root r has multiplicity k if the linear factor $(x-r)$ appears k-many times in the complete factorization of $P(x)$.

As a matter of fact, multiplicity shows up in the shape of the graph near the x-axis:

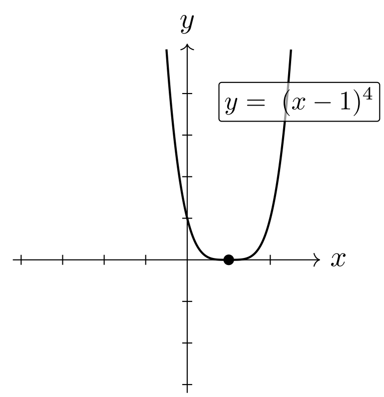
- if k is **odd** (for example $k = 1$), the graph typically **crosses** the x-axis at $x = r$,
- if k is **even** (for example $k = 2$), the graph **touches** the x-axis at $x = r$ and turns around.



$r = 1$ **has multiplicity** 2



$r = 1$ **has multiplicity** 3



$r = 1$ **has multiplicity** 4

Solutions to the Exercises

Exercise 1

1. $x^2 - 36 = (x+6)(x-6)$.
2. $x^2 - \dfrac{4}{25} = x^2 - \left(\dfrac{2}{5}\right)^2 = \left(x + \dfrac{2}{5}\right)\left(x - \dfrac{2}{5}\right)$.
3. $81x^2 - 49 = (9x)^2 - 7^2 = (9x+7)(9x-7)$.
4. $9a^2 - 16b^2 = (3a)^2 - (4b)^2 = (3a+4b)(3a-4b)$.
5. $25y^4 - 1 = (5y^2)^2 - 1^2 = (5y^2+1)(5y^2-1)$.

Exercise 2

1. $x^4 - 1 = (x^2+1)(x^2-1) = (x^2+1)(x+1)(x-1)$.

2. $x^4 - 16 = (x^2)^2 - 4^2 = (x^2+4)(x^2-4) = (x^2+4)(x+2)(x-2)$.
3. $20x^3 - 5x = 5x(4x^2-1) = 5x(2x+1)(2x-1)$.
4. $18y^2 - 2 = 2(9y^2-1) = 2(3y+1)(3y-1)$.
5. $8x^3 + 27 = (2x)^3 + 3^3 = (2x+3)(4x^2-6x+9)$.

Exercise 3

1. $m^2 - 4m + 4 = (m-2)^2$ (perfect square trinomial).
2. Not perfect square (middle term would need to be $\pm 4x$).
3. Not perfect square (last term is negative, not a square).
4. $x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2$ (perfect square trinomial).
5. $9p^2 + 12pq + 4q^2 = (3p+2q)^2$ (perfect square trinomial).

Exercise 4

1. $y^3 + 27 = (y+3)(y^2-3y+9)$.
2. $64 - x^3 = (4-x)(16+4x+x^2)$.
3. $2x^3 - 16 = 2(x^3-8) = 2(x-2)(x^2+2x+4)$.
4. $a^3 - 8 = (a-2)(a^2+2a+4)$.
5. $27m^3 + 1 = (3m)^3 + 1^3 = (3m+1)(9m^2-3m+1)$.

Exercise 5

1. $x^2 - 2x + 16 = 6x \Rightarrow x^2 - 8x + 16 = 0 \Rightarrow (x-4)^2 = 0$ so $x = 4$.
2. $3x^3 = 15x^2 + 18x \Rightarrow 3x^3 - 15x^2 - 18x = 0 \Rightarrow 3x(x^2-5x-6) = 0 \Rightarrow 3x(x-6)(x+1) = 0$ so $x = 0, 6, -1$.

Exercise 6

1. $x^2 - 4x + 4 = (x-2)^2 = 0$ so $x = 2$.
2. $x^2 + 3x + 17 = 1 - 7x \Rightarrow x^2 + 10x + 16 = 0 \Rightarrow (x+8)(x+2) = 0$ so $x = -8$ or $x = -2$.
3. $x^3 - 7x^2 + 10x = x(x^2-7x+10) = x(x-5)(x-2) = 0$ so $x = 0, 5, 2$.
4. $5x^3 + 33x^2 + 90x = x^3 - 3x^2 + 10x \Rightarrow 4x^3 + 36x^2 + 80x = 0 \Rightarrow 4x(x^2+9x+20) = 0 \Rightarrow 4x(x+5)(x+4) = 0$ so $x = 0, -5, -4$.
5. No real solutions (since $x^2 + 1 > 0$ for all real x).