MAT140 — Lecture 11 Handout
Rational Expressions

Throughout our discussion of polynomials, we have been discovering a rhyme:

$$\text{polynomials} \longleftrightarrow \text{numbers}.$$

We can add, subtract, multiply, divide, and simplify polynomials in ways that closely resemble what we do with ordinary numbers. As a matter of fact, our rhyme is slightly more nuanced: what we have been demonstrating so far is that polynomials are not just “numbers” in some vague sense – they are actually much like the *integers*.

In this lecture and the next, we finish our discussion of this rhyme by describing an analogue of *rational numbers* for polynomials. Recall that usual rational numbers are simply those numbers we can write in the form of a ratio, something like $\dfrac{a}{b}$ where a and b are integers. We will now repeat this idea, but we will take fractions whose numerator and denominator are *polynomials*, for example:

$$\frac{x^2+3}{x-2}.$$

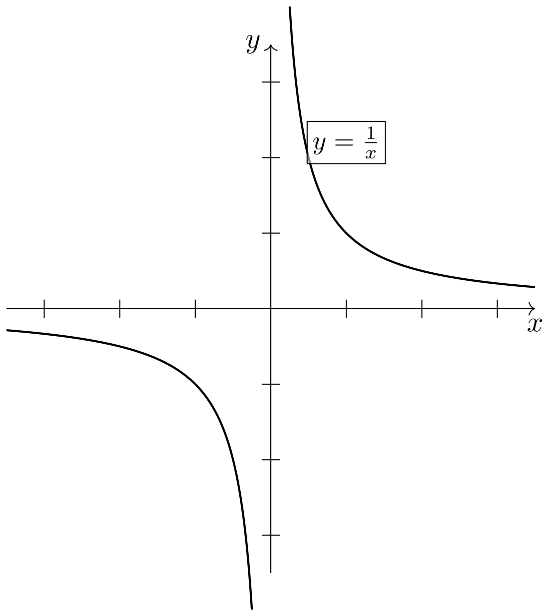
These are also called **rational expressions**, and when we treat them as functions of x we sometimes call them **rational functions**. As we will see, the manipulation of these rational expressions closely mimics the standard rules of fractions that you are familiar with. There are, of course, a few subtleties, but generally these rational expressions are somewhat intuitive.

Today we will:

1. Define a notion of “polynomial fractions”, and learn how to find their **domain**.
2. Simplify these new fractions by factoring and cancelling common factors (while keeping the correct domain restrictions).
3. Multiply, divide, add, and subtract polynomial fractions, using the least common denominator when needed.

1 Why division by zero is not allowed

A normal fraction is not well-defined if its denominator is 0. For example, $\frac{1}{0}$ is not a real number. One way to explain this is graphically:

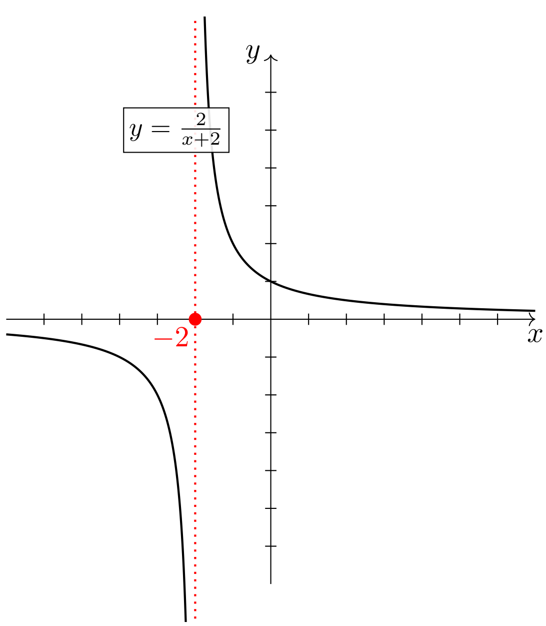


As you can see, when we let x approach zero from the right, the number $y = \dfrac{1}{x}$ gets bigger and bigger. When we let x approach zero from the left, the number gets bigger and bigger in the opposite direction. This is not good for several reasons, and ultimately we must conclude that at zero itself, the value of $\frac{1}{0}$ cannot be described by any real number.

The same idea applies to expressions with variables. For example, the expression

$$\frac{2}{x+2} \text{ is not well-defined when } x = -2.$$

Graphically, this is obvious:



Generally speaking, fractions are problematic whenever the denominator is zero. Since a variable like x can values across the entire real number line, there can be situations in which denominators involving variables can accidentally become zero at different values of x. In the previous example, we would have:

$$\frac{1}{x+2} \text{ at } x = -2 \text{ becomes: } \frac{1}{(-2)+2} = \frac{1}{0}$$

which is undefined.

Exercise 1

For each expression, state the values of x that make it not well-defined.

1. $\frac{15}{x}$
2. $\frac{15}{x+2}$
3. $\frac{9}{4-2x^2}$

2 Rational Expressions, Rational Functions, and Domain

We now introduce some precise terminology that will allows us to talk about fractions formed from polynomials.

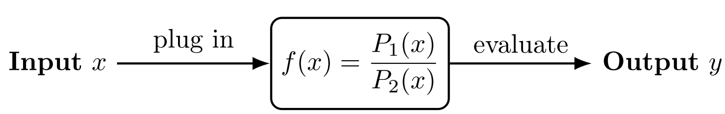
Rational expression

A **rational expression** is an expression of the form

$$\frac{P_1(x)}{P_2(x)}$$

where $P_1(x)$ and $P_2(x)$ are polynomials and $P_2(x) \neq 0$.

Sometimes, it is more useful to imagine a rational expression as defining a *function*. A rational expression like $\frac{P_1(x)}{P_2(x)}$ is a symbolic object that we can manipulate and simplify, but a rational *function* lets us view the expression as more of a rule with inputs and outputs:



Domain: all real x such that $P_2(x) \neq 0$

The "function" viewpoint turns a fraction of polynomials into something we can evaluate, graph, and potentially use in applications.

Rational function and domain

A **rational function** is a function of the form

$$f(x) = \frac{P_1(x)}{P_2(x)}$$

where $P_1(x)$ and $P_2(x)$ are polynomials.
Its **domain** is the set of all real numbers x for which the output is well-defined, meaning

$$P_2(x) \neq 0.$$

So the domain is: **all real numbers except the ones that make the denominator equal to 0.**

How to find the domain of a rational function

To find the domain of $f(x) = \dfrac{P_1(x)}{P_2(x)}$:

1. Set the denominator equal to zero: $P_2(x) = 0$.
2. Solve that equation.
3. Exclude those values from the domain.

Example

$$f(x) = \frac{4}{x-2}.$$

The denominator is zero when $x - 2 = 0$, meaning $x = 2$. So the domain is all real numbers **except** $x = 2$.

$$g(x) = \frac{5x}{x^2 - 16}.$$

The denominator is zero when $x^2 - 16 = 0$, meaning $(x-4)(x+4) = 0$. So the denominator is zero when $x = 4$ or $x = -4$. Thus the domain is all real numbers except $x = \pm 4$.

Exercise 2

Find the domain of each rational function.

1. $h(x) = \dfrac{x+1}{x^2-9}$
2. $k(x) = \dfrac{2x-5}{x^2+4x}$

3 Simplifying Rational Expressions

With number fractions, simplification means cancelling common factors:

$$\frac{15}{25} = \frac{3 \cdot 5}{5 \cdot 5} = \left(\frac{3}{5}\right) \cdot \left(\frac{5}{5}\right) = \left(\frac{3}{5}\right) \cdot (1) = \frac{3}{5},$$

which is often abbreviated with the “slash” notation:

$$\frac{15}{25} = \frac{3 \cdot 5}{5 \cdot 5} = \frac{3 \cdot \cancel{5}}{5 \cdot \cancel{5}} = \frac{3}{5},$$

The same idea works for polynomials, but first you usually need to factorize.

Simplifying rational expressions

To simplify

$$\frac{P_1(x)}{P_2(x)},$$

1. Factor $P_1(x)$ completely.
2. Factor $P_2(x)$ completely.
3. Cancel out any common factors.
4. State the domain restriction(s): any value that makes the *original* denominator zero is still excluded.

It is very important to note that this cancellation technique only works when the denominator is non-zero. In other words, we cannot “cancel out” copies of zero, i.e. writing something like:

$$\frac{3 \cdot \cancel{0}}{\cancel{0}} = 3 \textbf{ is not correct}.$$

Recall that the “cancellation” notation is shorthand for dividing a common factor by itself. So, if we could (incorrectly) cancel 0’s like any other number, this would mean that:

$$3 = \frac{3 \cdot 0}{0} = (3) \cdot \left(\frac{0}{0}\right)$$

which implies that $\frac{0}{0} = 1$, which is not true. Even worse, if $\frac{0}{0}$ *were* equal to 1, then any two numbers would be equal. For example, since $3 \cdot 0 = 0$, we could conclude:

$$3 = \frac{3 \cdot 0}{0} = \frac{0}{0} = 1$$

which is an obvious disaster. Hopefully this convinces you that we cannot cancel out zeroes in the denominator of fractions.

Let’s now consider an example of simplification of rational expressions. Consider the expression $\frac{x^2-1}{3x-3}$. We can follow the procedure above by first factorizing the numerator/denominator and then cancelling where possible:

$$\frac{x^2-1}{3x-3} = \frac{(x+1)(x-1)}{3(x-1)} = \frac{x+1}{3}, \qquad \text{where } x \neq 1.$$

Even though the simplified expression is $\frac{x+1}{3}$, the original fraction was undefined at $x = 1$, so we keep the restriction $x \neq 1$. Again, to reiterate, at the point $x = 1$ the above algebra **does not work**, because we would be doing something like:

$$\frac{(1+1)(1-1)}{3(1-1)} = \frac{2 \cdot \cancel{0}}{3 \cdot \cancel{0}} \neq \frac{2}{3}$$

which is not allowed, for the same reasons mentioned previously. So, in order to avoid this exact situation, we need to specify that $x \neq 1$.

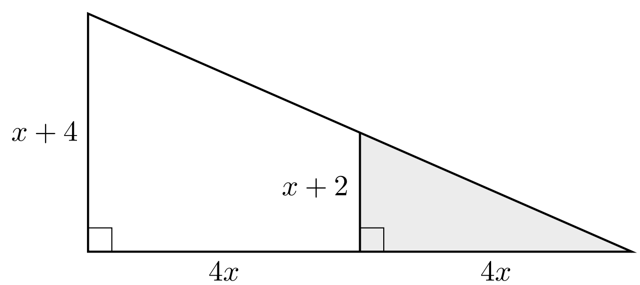
Exercise 3

Simplify each rational expression, and state any domain restriction(s).

1. $\dfrac{x^2 - 1}{3x - 3}$
2. $\dfrac{2x^3 - 6x^2}{6x^2}$
3. $\dfrac{9x^2 - 4}{3x - 2}$

3.1 Application: a geometry ratio

Suppose that we have a right-angled triangle with base length $4x + 4x = 8x$ and height $x + 4$. The smaller right-angle triangle shaded below has base $4x$ and height $x + 2$.



Suppose now that we want to find the ratio

$$\frac{\text{Area of small triangle}}{\text{Area of big triangle}}, \quad \text{assuming } x > 0.$$

How would we do this? Firstly, we should recall the formula for the area of a triangle:

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}).$$

Applying this formula twice, we have:

$$\text{Small area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(4x)(x+2)$$

$$\text{Large area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(8x)(x+4)$$

putting these two together, we obtain the following rational expression, which we simplify:

$$\frac{\left(\frac{1}{2}\right)(4x)(x+2)}{\left(\frac{1}{2}\right)(8x)(x+4)} = \frac{\left(\frac{1}{2}\right)(4x)(x+2)}{\left(\frac{1}{2}\right)(2)(4x)(x+4)} = \frac{\cancel{\left(\frac{1}{2}\right)}\cancel{(4x)}(x+2)}{\cancel{\left(\frac{1}{2}\right)}(2)\cancel{(4x)}(x+4)} = \frac{x+2}{2(x+4)} = \frac{x+2}{2x+8},$$

where here $x > 0$.

Exercise 4

A large right triangle has base $10x$ and height $x+5$. A smaller right triangle has base $4x$ and height $x+2$. Find the ratio

$$\frac{\text{Area of small triangle}}{\text{Area of big triangle}}, \quad \text{assuming } x > 0,$$

and simplify your answer.

4 Multiplying Rational Expressions

Recall that we multiply two fractions by multiplying the numerators together and multiplying the denominators together. In fact, multiplying rational expressions works *exactly* like multiplying normal fractions. We have:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

where here a, b, c and d are algebraic expressions, provided the denominators are not zero. A step-by-step process is summarised below.

Multiplying rational expressions

To multiply rational expressions:

1. Multiply the numerators together.
2. Multiply the denominators together.
3. Factor and simplify if possible.
4. State domain restrictions: exclude values that make any original denominator zero.

For example, to multiply the expressions $\frac{x^2-3}{3x}$ and $\frac{3}{2x}$ together, we simply treat them just like regular fractions:

$$\frac{x^2-3}{3x} \cdot \frac{3}{2x} = \frac{3(x^2-3)}{6x^2} = \frac{x^2-3}{2x^2}, \qquad \text{where } x \neq 0.$$

Exercise 5

Multiply and simplify. State domain restrictions.

1. $\dfrac{x^2-9}{x^2-4x} \cdot \dfrac{x}{x+3}$
2. $\dfrac{4x^3y}{3xy^4} \cdot \dfrac{-6x^2y^2}{10x^4}$

5 Dividing Rational Expressions

Since division is just the opposite operation of multiplication, to divide one fraction by another, we simply flip invert the second fraction and perform a multiplication. For example:

$$\frac{1}{3} \div \frac{2}{3} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1 \cdot 3}{2 \cdot 3} = \frac{1 \cdot \cancel{3}}{2 \cdot \cancel{3}} = \frac{1}{2}.$$

Dividing fractions uses the reciprocal:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Dividing rational expressions

To divide rational expressions:

1. Rewrite division as multiplication by the reciprocal (flip the second rational expression upside-down).
2. Multiply the two expressions that you have just written.
3. Simplify by removing common factors from the numerator/denominator, where possible.
4. Domain restrictions:
   - Exclude any value that makes any original denominator zero.
   - Also exclude any value that makes the *entire divisor* equal to 0.

As an example, let's divide the rational expression $\frac{x}{x+3}$ by $\frac{4}{x-1}$. Using the above step-by-step process, we have:

$$\frac{x}{x+3} \div \frac{4}{x-1} = \frac{x}{x+3} \cdot \frac{x-1}{4} = \frac{x(x-1)}{4(x+3)} = \frac{x^2-x}{4x+12}, \qquad \text{where } x \neq -3,\ x \neq 1.$$

Based on the example above, we see that division is really just another form of multiplication. There is, however, one subtle difference: item (4) above tells us to include some extra potential restrictions when dividing. To make this a bit more clear, let's use an example: suppose that we try to divide $\frac{x^2}{x+1}$ by $\frac{x-1}{x+3}$. Ultimately, we are trying to avoid those values of x that would accidentally cause us to divide by zero. There are two obvious candidates: the denominators of both fractions always need to be non-zero. However, here there is a third possible problem case: when the *entire expression* $\frac{x-1}{x+3}$ equals zero. This is because we are trying to perform a division, so:

$$\text{if } \frac{x-1}{x+3} = 0,\ \text{then } \frac{x^2}{x+1} \div \frac{x-1}{x+3} \text{ becomes } \frac{x^2}{x+1} \div 0,\ \text{which is undefined.}$$

It is possible for a rational expression to be zero whenever the *numerator* is zero. In this case, we have $\frac{x-1}{x+3} = 0$ whenever $x - 1 = 0$ and $x + 3 \neq 0$. So, when stating our restrictions for this division, we also need to state that $x \neq 1$.

Exercise 6

Divide and simplify. State domain restrictions.

1. $\dfrac{x}{x+3} \div \dfrac{4}{x-1}$
2. $\dfrac{2x}{3x-12} \div \dfrac{x^2-2x}{x^2-6x+8}$

6 Adding and Subtracting Rational Expressions

6.1 Like denominators

If two fractions have matching denominators, then their sum is particularly simple: we simply add the numerators together. For example:

$$\frac{1}{5} + \frac{3}{5} = \frac{4}{5}.$$

Addition of rational expressions works the same way.

Adding/subtracting with like denominators

Let a, b and c be algebraic expressions. If $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \qquad \text{and} \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

Exercise 7

Simplify each expression and state domain restrictions.

1. $\dfrac{x}{4} + \dfrac{5-x}{4}$
2. $\dfrac{7}{2x-3} - \dfrac{3x}{2x-3}$
3. $\dfrac{x}{x^2-2x-3} - \dfrac{3}{x^2-2x-3}$

6.2 Unlike denominators, LCD via LCM

If the denominators of two fractions do not match, then we first "unsimplify" the pair into equivalent fractions that *do* have matching denominators. For example:

$$\frac{1}{4} + \frac{1}{3} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}.$$

Typically, we can find new matching denominators by multiplying the two starting denominators together. In the case above, $3 \times 4 = 12$. However, this is not always necessary. For example, to compute the sum

$$\frac{7}{60} + \frac{19}{120},$$

it is not necessary to convert the fractions into things with denominators 60×120. Instead, we can simply observe that $60 \times 2 = 120$, and therefore we would just convert the first fraction into something over 120:

$$\frac{7}{60} + \frac{19}{120} = \frac{14}{120} + \frac{19}{120} = \frac{23}{120}.$$

Here 120 is the *least common multiple* of 60 and 120, that is, it is the smallest number that has both 60 and 120 as factors.

Least common multiple (LCM) for polynomials

The **least common multiple** of two or more polynomials is the simplest polynomial that is a multiple of each one.
How to find it:

1. Factor each polynomial completely.
2. Collect every distinct factor that appears.
3. For each factor, use the highest power it appears with in any polynomial.

In other words, the least common multiple of a collection of polynomials $P_1(x)$ and $P_2(x)$ is the "smallest" polynomial that has both $P_1(x)$ and $P_2(x)$ as factors.[1] In practice, there are a few useful situations which will help determine what the LCM of a pair of polynomials is.

Useful Tricks for Finding LCMs

Let $P_1(x)$ and $P_2(x)$ be polynomials.

1. If $P_1(x)$ divides $P_2(x)$ evenly, then the LCM is equal to $P_2(x)$.
2. If $P_2(x)$ divides $P_1(x)$ evenly, then the LCM is equal to $P_1(x)$.
3. If $P_1(x) \neq P_2(x)$ and both are irreducible, then their LCM is the product $P_1(x)P_2(x)$.

As a matter of fact, these tricks emerge from a more general rule:

$$LCM(P_1, P_2) = \frac{P_1(x) \cdot P_2(x)}{GCD(P_1, P_2)},$$

where here GCD is the greatest common divisor of the polynomials – the largest polynomial that evenly divides both $P_1(x)$ and $P_2(x)$. We will now present a couple of examples.

1. Consider first the polynomials $x + 1$ and $x^2 - 1$. Here, notice that we can factorize the latter to get $(x + 1)(x - 1)$. In this form, we see that $x + 1$ is obviously a divisor of $x^2 - 1$. Therefore, we can apply rule (1) in the red box above to conclude that the least common multiple of $x + 1$ and $x^2 - 1$ is $x^2 - 1$ itself.
2. Consider now the polynomials $x + 1$ and $x^2 + 1$. In last lecture, we saw that $x^2 + 1$ is actually an irreducible polynomial, since its graph does not cross the x-axis. The polynomial $x + 1$ is also clearly irreducible, and therefore we may apply rule (3) in the red box above to conclude that the least common multiple of $x + 1$ and $x^2 + 1$ is their product $(x + 1)(x^2 + 1)$.

Exercise 8

Find the least common multiple of the expressions.

1. $6x,\ 2x^2,\ 9x^3$
2. $x^2 - x,\ 2x - 2$
3. $3x^2 + 6x,\ x^2 + 4x + 4$

As with our earlier example of $\frac{7}{60} + \frac{19}{120}$, the addition of rational expressions can be performed by looking for least common multiples.

[1] Another way to say this: the polynomials $P_1(x)$ and $P_2(x)$ both divide their LCM evenly, and the LCM is the simplest polynomial with this property.

Strategy for adding/subtracting with unlike denominators

To add or subtract rational expressions with unlike denominators:

1. Factor denominators.
2. Find the least common denominator (the LCM of the denominators).
3. Rewrite each fraction with the least common denominator.
4. Add or subtract the numerators, keep the least common denominator.
5. Simplify if possible and state domain restrictions.

As an example, let's consider the addition:

$$\frac{7}{6x}+\frac{5}{8x}.$$

Here, we can see that the two denominators are not the same, so we will need to use their least common multiple to convert the fractions to a pair that have the same denominator. There are (at least) two ways to do this:

1. we notice that both polynomials have the same power of x, with different coefficients. So, if we find the LCM of 6 and 8 then we can simply add an x to create the LCM of $6x$ and $8x$. By process of elimination, we can notice that the LCM of 6 and 8 is 24.[2] Therefore, the LCM of $6x$ and $8x$ will be $24x$.
2. we may use the formula $\text{LCM} = \frac{\text{product}}{\text{GCD}}$. In this case, the greatest common divisor of $6x$ and $8x$ is $2x$, and therefore the least common multiple will be $\frac{(6x)(8x)}{2x} = \frac{48x^2}{2x} = 24x$.

Through either method, we see that the LCD should be $24x$. Now, to complete the setup, we need to convert both these fractions. In order to do this, we multiply each fraction by the "missing piece" that will yield the denominator of $24x$. For the fraction with denominator $6x$, we notice that we need to multiply it by 4, and for the fraction with denominator $8x$, we notice that we need to multiply it by 3. Once we do this conversion we can then add the results easily:

$$\frac{7}{6x}+\frac{5}{8x}=\frac{7(4)}{6x(4)}+\frac{5(3)}{8x(3)}=\frac{28}{24x}+\frac{15}{24x}=\frac{43}{24x}, \qquad \text{where } x \neq 0.$$

Exercise 9

Add and simplify. State domain restrictions.

1. $\frac{1}{2x}+\frac{3x}{4}$
2. $\frac{x+2}{2}+\frac{7}{2x}$
3. $\frac{x-2}{3}+\frac{5}{x}$
4. $\frac{1}{6x^2}+\frac{3}{x}$
5. $\frac{2x}{x^2-1}+\frac{1}{x+1}$
6. $\frac{2x+2}{x^2-3x-4}+\frac{3}{x-4}$

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[2] If you want to be more precise here: $6 = 2 \times 3$ and $8 = 2 \times 2 \times 2$, so the least common multiple will be $2 \times 2 \times 2 \times 3 = 24$.

Solutions to the Exercises

Exercise 1

1. Not well-defined when $x = 0$.
2. Not well-defined when $x + 2 = 0$, meaning $x = -2$.
3. Not well-defined when $4 - 2x^2 = 0$. Solving: $2x^2 = 4$, so $x^2 = 2$, so $x = \pm\sqrt{2}$.

Exercise 2

1. $h(x) = \dfrac{x+1}{x^2-9}$ has denominator 0 when $x^2 - 9 = 0$, meaning $x = \pm 3$. Domain: all real x such that $x \neq -3$ and $x \neq 3$.
2. $k(x) = \dfrac{2x-5}{x^2+4x}$ has denominator 0 when $x^2 + 4x = 0$, meaning $x(x+4) = 0$. Domain: all real x such that $x \neq 0$ and $x \neq -4$.

Exercise 3

1. $$\frac{x^2-1}{3x-3} = \frac{(x+1)(x-1)}{3(x-1)} = \frac{x+1}{3}, \qquad \text{where } x \neq 1.$$
2. $$\frac{2x^3-6x^2}{6x^2} = \frac{2x^2(x-3)}{6x^2} = \frac{x-3}{3}, \qquad \text{where } x \neq 0.$$
3. $$\frac{9x^2-4}{3x-2} = \frac{(3x-2)(3x+2)}{3x-2} = 3x+2, \qquad \text{where } x \neq \frac{2}{3}.$$

Exercise 4

$$\frac{\text{Area of small triangle}}{\text{Area of big triangle}} = \frac{\left(\frac{1}{2}\right)(4x)(x+2)}{\left(\frac{1}{2}\right)(10x)(x+5)} = \frac{4x(x+2)}{10x(x+5)} = \frac{2(x+2)}{5(x+5)}, \qquad \text{where } x > 0.$$

Exercise 5

1. $$\frac{x^2-9}{x^2-4x} \cdot \frac{x}{x+3} = \frac{(x-3)(x+3)}{x(x-4)} \cdot \frac{x}{x+3} = \frac{x-3}{x-4},$$
   where $x \neq 0$ and $x \neq 4$ (from $x^2 - 4x$) and $x \neq -3$ (from $x + 3$).
2. $$\frac{4x^3y}{3xy^4} \cdot \frac{-6x^2y^2}{10x^4} = \frac{-24x^5y^3}{30x^5y^4} = -\frac{4}{5y}, \qquad \text{where } x \neq 0,\ y \neq 0.$$

Exercise 6

1. $$\frac{x}{x+3} \div \frac{4}{x-1} = \frac{x}{x+3} \cdot \frac{x-1}{4} = \frac{x(x-1)}{4(x+3)}, \qquad \text{where } x \neq -3,\ x \neq 1.$$

2. $$\frac{2x}{3x-12} \div \frac{x^2-2x}{x^2-6x+8} = \frac{2x}{3(x-4)} \cdot \frac{(x-2)(x-4)}{x(x-2)} = \frac{2}{3}.$$

   Domain restrictions: $x \neq 4$ (from $3x-12$), $x \neq 2, 4$ (from x^2-6x+8), and also the divisor cannot be 0, so $x^2-2x \neq 0$, meaning $x \neq 0, 2$. Combined: $x \neq 0, 2, 4$.

Exercise 7

1. $$\frac{x}{4} + \frac{5-x}{4} = \frac{x+5-x}{4} = \frac{5}{4}.$$

2. $$\frac{7}{2x-3} - \frac{3x}{2x-3} = \frac{7-3x}{2x-3}, \qquad \text{where } x \neq \frac{3}{2}.$$

3. Factor $x^2-2x-3 = (x-3)(x+1)$:

   $$\frac{x}{x^2-2x-3} - \frac{3}{x^2-2x-3} = \frac{x-3}{x^2-2x-3} = \frac{x-3}{(x-3)(x+1)} = \frac{1}{x+1},$$

   where $x \neq 3$ and $x \neq -1$.

Exercise 8

1. $6x = 2 \cdot 3 \cdot x$, $2x^2 = 2 \cdot x^2$, $9x^3 = 3^2 \cdot x^3$. So LCM $= 2 \cdot 3^2 \cdot x^3 = 18x^3$.
2. $x^2 - x = x(x-1)$ and $2x-2 = 2(x-1)$. So LCM $= 2x(x-1)$.
3. $3x^2+6x = 3x(x+2)$ and $x^2+4x+4 = (x+2)^2$. So LCM $= 3x(x+2)^2$.

Exercise 9

1. $$\frac{1}{2x} + \frac{3x}{4} = \frac{2}{4x} + \frac{3x^2}{4x} = \frac{3x^2+2}{4x}, \qquad \text{where } x \neq 0.$$

2. $$\frac{x+2}{2} + \frac{7}{2x} = \frac{x(x+2)}{2x} + \frac{7}{2x} = \frac{x^2+2x+7}{2x}, \qquad \text{where } x \neq 0.$$

3. $$\frac{x-2}{3} + \frac{5}{x} = \frac{x(x-2)}{3x} + \frac{15}{3x} = \frac{x^2-2x+15}{3x}, \qquad \text{where } x \neq 0.$$

4. $$\frac{1}{6x^2} + \frac{3}{x} = \frac{1}{6x^2} + \frac{18x}{6x^2} = \frac{1+18x}{6x^2}, \qquad \text{where } x \neq 0.$$

5. Factor $x^2-1 = (x-1)(x+1)$:

   $$\frac{2x}{x^2-1} + \frac{1}{x+1} = \frac{2x}{(x-1)(x+1)} + \frac{x-1}{(x-1)(x+1)} = \frac{3x-1}{(x-1)(x+1)}, \qquad \text{where } x \neq 1,\ x \neq -1.$$

6. Factor $x^2 - 3x - 4 = (x-4)(x+1)$ and $2x+2 = 2(x+1)$:

$$\frac{2x+2}{x^2-3x-4} + \frac{3}{x-4} = \frac{2(x+1)}{(x-4)(x+1)} + \frac{3}{x-4} = \frac{2}{x-4} + \frac{3}{x-4} = \frac{5}{x-4}.$$

Domain restrictions come from the original denominators: $x \neq 4$ and $x \neq -1$.