MAT140 — Lecture 12 Handout

More on Rational Expressions

Last lecture we worked with **polynomial fractions** (also called **rational expressions**). We learned how to simplify them by factorizing, and how to multiply and divide them by using factorization and cancellation.

In this lecture we push these ideas further in two ways. First, we learn how to simplify **complex fractions**, which are fractions built out of smaller fractions. Second, we learn how to solve certain **rational equations** by clearing denominators. Finally, we see a handful of short applications where simple algebraic models give useful predictions (linear, quadratic, and inverse-square models).

Today we will:

1. Review simplification, multiplication, division, and addition of rational expressions.
2. Learn how to simplify **complex fractions**.
3. Learn how to solve **rational equations** by multiplying by a least common denominator.
4. Practice algebraic models in context (saving money, temperature conversion, trajectories, inverse square law).

1 Complex Fractions

Problems involving the division of two rational expressions are sometimes written as **complex fractions**.

Complex fraction

A **complex fraction** is a fraction that has a fraction in its numerator or denominator (or both).

For example, if $\frac{P_1}{P_2}$ and $\frac{P_3}{P_4}$ are polynomial fractions, then a complex fraction built from these could be something of the form:

$$\frac{\frac{P_1}{P_2}}{\frac{P_3}{P_4}}.$$

Based on the way that it is presented, it's not entirely clear how to unpack a complex fraction to simplify it. Perhaps the easiest way is to remember that fractions represent a *division*. We can insert parentheses and emphasize the dominant division to understand how to simplify:

$$\frac{\frac{P_1}{P_2}}{\frac{P_3}{P_4}} = \frac{\left(\frac{P_1}{P_2}\right)}{\left(\frac{P_3}{P_4}\right)} = \left(\frac{P_1}{P_2}\right) \div \left(\frac{P_3}{P_4}\right) = \frac{P_1}{P_2} \times \frac{P_4}{P_3} = \frac{P_1 \cdot P_4}{P_2 \cdot P_3}.$$

This step-by-step process is summarized below.

How to simplify a complex fraction

To simplify a complex fraction:

1. **Write it out** as a division problem.
2. **Flip** the denominator fraction (take the reciprocal).
3. **Multiply** the numerator by that reciprocal.
4. **Simplify** by factorizing and cancelling common factors.

To see this process in action, let's consider a complex fraction of rational numbers:

$$\frac{\frac{5}{14}}{\frac{25}{8}}.$$

$$\frac{\frac{5}{14}}{\frac{25}{8}} = \frac{\left(\frac{5}{14}\right)}{\left(\frac{25}{8}\right)} = \frac{5}{14} \div \frac{25}{8} = \frac{5}{14} \cdot \frac{8}{25} = \frac{5 \cdot 8}{14 \cdot 25} = \frac{5 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 7 \cdot 5 \cdot 5} = \frac{4}{35}.$$

To simplify complex fractions involving polynomials, we can follow the same pattern, this time using polynomial arithmetic instead of integer arithmetic. As an example, let's consider the complex fraction:

$$\frac{\frac{x+2}{3}}{\frac{x-2}{x}}.$$

This expression can be simplified as follows:

$$\frac{\frac{x+2}{3}}{\frac{x-2}{x}} = \frac{x+2}{3} \div \frac{x-2}{x} = \frac{x+2}{3} \cdot \frac{x}{x-2} = \frac{x(x+2)}{3(x-2)}.$$

Since this is now a rational expression involving polynomials, there is a chance that we might accidentally let x take a value that would cause us to divide by zero. So, we need to specify the problematic values of x and make sure to exclude them. In this case, there are two problematic values:

- if $x = 0$ then the second fraction $\frac{x-2}{x}$ is not well defined.
- if $x = 2$ then the second fraction $\frac{x-2}{x}$ would equal zero, meaning that the complex fraction would become $\frac{x+2}{3} \div 0$, which is again undefined.

So, our restrictions are: $x \neq 0$ and $x \neq 2$.

Common mistakes

- Do not cancel *across addition or subtraction*. You can only cancel **factors** in a product.
- Always simplify by **factorizing first**, then cancel common factors.
- Keep track of **restrictions** from the original denominators.

Exercise 1

Simplify each complex fraction. State any restrictions on x.

1. $\frac{\frac{x+1}{4}}{\frac{x-1}{2}}$

2. $\dfrac{\frac{2x+1}{x}}{\frac{x-1}{x+1}}$
3. $\dfrac{\frac{x-1}{x^2-1}}{\frac{2}{x+1}}$

1.1 Awkward complex fractions

Sometimes a complex fraction is written in a form that is harder to read and simplify at first. In that case, the best move is usually to rewrite the expressions in a nicer form first, and then deal with the division afterwards.

As an example, consider the complex fraction:

$$\frac{\frac{x}{3}+\frac{2}{3}}{1-\frac{2}{x}}.$$

To simplify this, it's easier to first rewrite the denominator:

$$1-\frac{2}{x}=\frac{x}{x}-\frac{2}{x}=\frac{x-2}{x}.$$

Similarly, we can also rewrite the numerator:

$$\frac{x}{3}+\frac{2}{3}=\frac{x+2}{3}.$$

This allows us to turn the original expression into something easier to handle:

$$\frac{\frac{x}{3}+\frac{2}{3}}{1-\frac{2}{x}}=\frac{\frac{x+2}{3}}{\frac{x-2}{x}}=\frac{x+2}{3}\cdot\frac{x}{x-2}=\frac{x(x+2)}{3(x-2)}.$$

Regarding the restrictions, here x cannot equal zero, because that would imply the fraction $\frac{2}{0}$ is undefined, and x also cannot equal 2, because that would cause the entire denominator to be $1-\frac{2}{2}=1-1=0$.

Exercise 2

Simplify each complex fraction. State any restrictions on x.

1. $\dfrac{\frac{x+1}{4}}{1-\frac{1}{x}}$
2. $\dfrac{\left(2x-\frac{3}{x^2}\right)}{\left(x-\frac{1}{x}\right)}$

2 Solving Rational Equations

A **rational equation** is an equation that contains fractions with variables in the denominator. The main idea is to clear the denominators by multiplying both sides by a least common denominator

(LCD).

To see an example of this, let's start with a linear equation involving fractions. Consider the equation:

$$\frac{3}{5} = \frac{x}{2} + 1.$$

Of course, one method to solve this equation is to subtract 1 from both sides and then multiply everything by 2. However, we could also remove all of the denominators all at once by multiplying everything by 10, which is the least common multiple of 2 and 5. We have:

$$10\left(\frac{3}{5}\right) = 10\left(\frac{x}{2} + 1\right) \quad \Rightarrow \quad 6 = 5x + 10 \quad \Rightarrow \quad -4 = 5x \quad \Rightarrow \quad x = -\frac{4}{5}.$$

Of course, we can double-check that this is correct by substituting this proposed solution back in to the original equation:

$$\frac{x}{2} + 1 = \frac{-4/5}{2} + 1 = -\frac{2}{5} + 1 = \frac{3}{5},$$

so $x = -\frac{4}{5}$ is indeed correct.

Of course, equations involving rational expressions don't need to be as simple as the previous example. In general, a step-by-step procedure may be the following.

Solving a rational equation

To solve a rational equation:

1. Write down the **restrictions** (values that make any denominator 0).
2. Find the **least common denominator** (LCD).
3. Multiply **each side** of the equation by the LCD.
4. Distribute, simplify, and solve the resulting equation.
5. **Check** your solution(s) in the original equation (this catches extraneous solutions).

As an example of a slightly more complicated equation, consider:

$$\frac{8}{3} = \frac{7}{x} - \frac{1}{3x}.$$

We observe first that the equation has the single restriction $x \neq 0$. Now, to clear the denominators we need to find the least common denominator of 3, x and $3x$. The obvious answer here is $3x$. So, we multiply both sides of the equation by $3x$ to remove all of the denominators:

$$3x\left(\frac{8}{3}\right) = 3x\left(\frac{7}{x} - \frac{1}{3x}\right) \quad \Rightarrow \quad 8x = 21 - 1$$

From here, we see that $x = \frac{20}{8} = \frac{5}{2}$.

Exercise 3

Solve each equation. State any restrictions and check your solution(s).

1. $\dfrac{2}{x} + \dfrac{1}{3} = 1$
2. $\dfrac{1}{x-2} + \dfrac{1}{x+2} = \dfrac{1}{2}$

3 Applications of Polynomials

3.1 Buying a dog

Suppose that you want to buy a new dog, which costs 350000 yen. You currently have 150000 yen saved, and you can save 20000 yen per week. How long will it take to save up the required money?

For a question like this, a simple model is:

$$\text{money saved} = \text{initial savings} + \text{weekly savings} \times \text{time}.$$

Which is a linear equation that can be solved directly:

$$350000 = 150000 + 20000t \quad \Rightarrow \quad 350 = 150 + 20t \quad \Rightarrow \quad 200 = 20t \quad \Rightarrow \quad t = 10.$$

As we see, it will take **10 weeks**.

Exercise 4

You want to buy a car that costs 480000 yen. You have 120000 yen saved, and you can save 30000 yen per week. How many weeks will it take?

3.2 Converting temperature

Temperature conversion

The relationship between Fahrenheit F and Celsius C is:

$$F = \frac{9}{5}C + 32.$$

For example, suppose we want to convert 52°F into Celsius.

$$52 = \frac{9}{5}C + 32 \quad \Rightarrow \quad 20 = \frac{9}{5}C \quad \Rightarrow \quad C = 20 \cdot \frac{5}{9} = \frac{100}{9} \approx 11.1.$$

So 52°F is about 11.1°C.

Conversely, 52 degrees Celsius would be:

$$F = \frac{9}{5} \cdot 52 + 32 = \frac{468}{5} + 32 = \frac{468}{5} + \frac{160}{5} = \frac{628}{5} = 125.6.$$

So 52°C is 125.6°F.

Exercise 5

1. Convert 68°F into Celsius.
2. Convert 20°C into Fahrenheit.

3.3 Calculating a trajectory

A common model for projectile motion is:

$$h(t) = h_0 + v_0 t - \frac{1}{2} g t^2,$$

where:

- $h(t)$ is height at time t,
- h_0 is initial height,
- v_0 is initial vertical velocity,
- g is acceleration due to gravity.

For simplicity, we will use the values $h_0 = 1$ and $g \approx 10$, so that:

$$h(t) = 1 + v_0 t - 5t^2.$$

Suppose now that you are a pirate who would like to fire a cannon into the air.

Exercise 6

Assume $v_0 = 40$ m/s. Let

$$h(t) = 1 + 40t - 5t^2.$$

1. What is the height after 3 seconds?
2. At what time(s) is the cannonball at height 61 m?
3. Suppose now that we do not know v_0. If we want the cannonball to reach a height of 100 m after 3 seconds, what should the initial velocity be?

4 The Inverse Square Law

Inverse square law

An inverse-square model has the form

$$y = \frac{a}{x^2},$$

where a is a constant.

A key feature is how the output scales with distance:

- If x doubles, then x^2 becomes 4 times larger, so y becomes **four times smaller**.

- If x triples, then x^2 becomes 9 times larger, so y becomes **nine times smaller**.

This is essentially a statement about geometry in three dimensions: if we take a sphere of radius x and draw a small square on it, then the surface area of this square will grow at a rate proportional to x^2 whenever we increase x. Rearranging this growth gives the formula:

$$y = \frac{a}{x^2} \quad \Rightarrow \quad x^2 = \frac{a}{y} \quad \Rightarrow \quad x = \sqrt{\frac{a}{y}}.$$

This model appears in many places, for example:

$$F = G\frac{m_1 m_2}{r^2} \qquad \text{(Newton's law of gravitation)}, \qquad F = k\frac{q_1 q_2}{r^2} \qquad \text{(Coulomb's law)}.$$

Generally, most physical situations in which we can model a pointlike source of waves propagating out into $3d$ space will follow an inverse square law.

4.1 The campfire

A simple temperature model (radiation-style) is:

$$T(d) = 15 + \frac{k}{d^2},$$

where d is distance from the fire, 15 is a baseline temperature, and k is a constant.

Exercise 7

Suppose that at $d = 1$ m the temperature is $55°$.

1. Find k.
2. What is the temperature at $d = 2$ m? How about at $d = 0.5$ m?
3. Suppose we want the temperature to be $45°$. How far away should we sit?

Solutions to the Exercises

Exercise 1

1. $\frac{\frac{x+1}{4}}{\frac{x-1}{2}} = \frac{x+1}{4} \cdot \frac{2}{x-1} = \frac{x+1}{2(x-1)} = \frac{x+1}{2x-2}$. Restriction: $x \neq 1$.
2. $\frac{\frac{2x+1}{x}}{\frac{x-1}{x+1}} = \frac{2x+1}{x} \cdot \frac{x+1}{x-1} = \frac{(2x+1)(x+1)}{x(x-1)}$. Restrictions: $x \neq 0$, $x \neq 1$, $x \neq -1$.
3. $\frac{\frac{x-1}{x^2-1}}{\frac{2}{x+1}} = \frac{x-1}{x^2-1} \cdot \frac{x+1}{2} = \frac{x-1}{(x-1)(x+1)} \cdot \frac{x+1}{2} = \frac{1}{2}$. Restrictions: $x \neq 1$, $x \neq -1$.

Exercise 2

1. $\frac{\frac{x+1}{4}}{1-\frac{1}{x}} = \frac{\frac{x+1}{4}}{\frac{x-1}{x}} = \frac{x+1}{4} \cdot \frac{x}{x-1} = \frac{x(x+1)}{4(x-1)}$. Restrictions: $x \neq 0$, $x \neq 1$.
2. Rewrite numerator and denominator:

$$2x - \frac{3}{x^2} = \frac{2x^3-3}{x^2}, \qquad x - \frac{1}{x} = \frac{x^2-1}{x}.$$

Then:

$$\frac{\left(2x - \frac{3}{x^2}\right)}{\left(x - \frac{1}{x}\right)} = \frac{\frac{2x^3-3}{x^2}}{\frac{x^2-1}{x}} = \frac{2x^3-3}{x^2} \cdot \frac{x}{x^2-1} = \frac{x(2x^3-3)}{x^2(x^2-1)} = \frac{2x^3-3}{x(x^2-1)} = \frac{2x^3-3}{x^3-x}.$$

Restrictions: $x \neq 0$, $x \neq 1$, $x \neq -1$.

Exercise 3

1. Restriction: $x \neq 0$.

$$\frac{2}{x} + \frac{1}{3} = 1 \quad \Rightarrow \quad \text{multiply by } 3x: \ 6 + x = 3x \quad \Rightarrow \quad 6 = 2x \quad \Rightarrow \quad x = 3.$$

Check: $\frac{2}{3} + \frac{1}{3} = 1$. Good.

2. Restrictions: $x \neq 2$, $x \neq -2$.

$$\frac{1}{x-2} + \frac{1}{x+2} = \frac{1}{2}.$$

LCD is $2(x-2)(x+2)$. Multiply through:

$$2(x+2) + 2(x-2) = (x-2)(x+2) \quad \Rightarrow \quad 2x + 4 + 2x - 4 = x^2 - 4$$

$$4x = x^2 - 4 \quad \Rightarrow \quad x^2 - 4x - 4 = 0.$$

Solve:

$$x = \frac{4 \pm \sqrt{16+16}}{2} = \frac{4 \pm \sqrt{32}}{2} = 2 \pm 2\sqrt{2}.$$

Both values satisfy the restrictions, so the solutions are $x = 2 + 2\sqrt{2}$ and $x = 2 - 2\sqrt{2}$.

Exercise 4

$$480000 = 120000 + 30000t \quad \Rightarrow \quad 480 = 120 + 30t \quad \Rightarrow \quad 360 = 30t \quad \Rightarrow \quad t = 12.$$

It will take **12 weeks**.

Exercise 5

1. $68 = \frac{9}{5}C + 32 \Rightarrow 36 = \frac{9}{5}C \Rightarrow C = 36 \cdot \frac{5}{9} = 20$. So 68°F is 20°C.
2. $F = \frac{9}{5} \cdot 20 + 32 = 36 + 32 = 68$. So 20°C is 68°F.

Exercise 6

1. $h(3) = 1 + 40(3) - 5(3^2) = 1 + 120 - 45 = 76$ m.
2. Set $h(t) = 61$:

$$1 + 40t - 5t^2 = 61 \Rightarrow -5t^2 + 40t - 60 = 0 \Rightarrow t^2 - 8t + 12 = 0 \Rightarrow (t-2)(t-6) = 0.$$

   So $t = 2$ s or $t = 6$ s.
3. We want $h(3) = 100$ with $h(t) = 1 + v_0 t - 5t^2$:

$$100 = 1 + 3v_0 - 5(9) = 1 + 3v_0 - 45 \Rightarrow 100 = 3v_0 - 44 \Rightarrow 3v_0 = 144 \Rightarrow v_0 = 48 \text{ m/s}.$$

Exercise 7

We have $T(d) = 15 + \dfrac{k}{d^2}$ and $T(1) = 55$.

1. $55 = 15 + \dfrac{k}{1^2} \Rightarrow k = 40$.
2. $T(2) = 15 + \dfrac{40}{2^2} = 15 + \dfrac{40}{4} = 15 + 10 = 25$. Also,

$$T(0.5) = 15 + \frac{40}{(0.5)^2} = 15 + \frac{40}{0.25} = 15 + 160 = 175.$$

3. $45 = 15 + \dfrac{40}{d^2} \Rightarrow 30 = \dfrac{40}{d^2} \Rightarrow d^2 = \dfrac{4}{3}$. Since distance is positive,

$$d = \sqrt{\frac{4}{3}} \approx 1.15 \text{ m}.$$