MAT140 — Lecture 13 Handout

Simultaneous Equations

Earlier in the course, we discussed the algebraic properties of *linear equations*, and studied their graphical properties. We will now discuss an extension to this idea by looking at *collections* of these equations. These collections of linear equations are known as "systems of linear equations", or sometimes "simultaneous equations". We are going to spend the next four lectures talking about this topic: we will see that by starting with these collections, a fascinating new story of structure, representation, and application will start to emerge. Later on, we will define and study matrices, which will lead us naturally into an introduction to *linear algebra*. For now, we will start our journey with a humble beginning: two linear equations with two unknown variables.

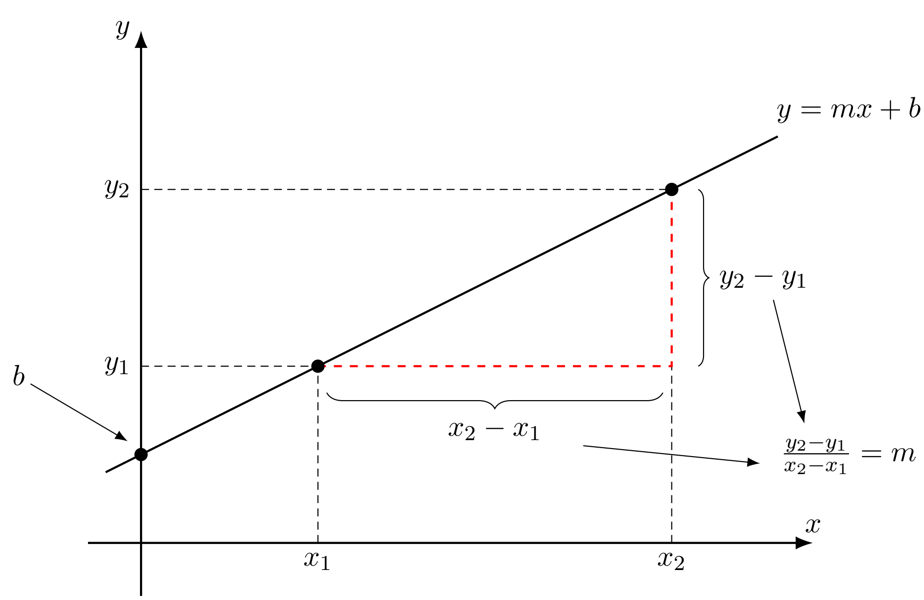
Today we will:

1. Define a **system of linear equations** and what it means to be a **solution**.
2. Solve a system by **graphing** and interpret solutions as intersections of lines.
3. Solve a system algebraically using **substitution**.
4. Solve a system algebraically using **elimination**.
5. Understand the "weird cases": **no solution** and **infinitely many solutions**.

1 Systems of Linear Equations and their Solutions

1.1 The Definition of a System of Linear Equations

A line is simply a curve that changes at a fixed, constant rate. As we have seen in previous lectures, this "rate" is known as a *gradient*, and it tells us how much the y-coordinate of a line grows according to changes in the x-coordinate. As well as this, we have also seen that all non-vertical lines have a unique point where they cross the y-axis – its so-called y-*intercept*. These two numbers uniquely specify the geometry of a line:



The resulting linear equation $y = mx + b$ describes a unique line in the coordinate plane. However, this is not the *only* description of the same line. In fact, there are many descriptions of the same geometric object. For example, here are three different descriptions of the same line:

$$y = 2x + 1 \qquad 3y = 6x + 3 \qquad -2x + y = 1.$$

In the cases of the latter two descriptions, we simply performed arithmetic operations balanced on both sides of the equation, so that the space of solutions to the equation remains preserved. The line itself, which is simply the set of solutions to any one of these equations, will be the same in all three cases.

In what follows, it will be very convenient to have our linear equations written in the form of the third example above. Such a description is called a linear equation in *standard form*.

Linear Equations in Standard Form

A linear equation in standard form is an equation

$$ay + bx = c,$$

where a, b and c are real numbers and x, y are variables, and we exclude the case that a and b are both zero at the same time.

Any linear equation in standard form can be written in gradient-intercept form using some basic algebraic manipulations:

$$ax + by = c \;\Rightarrow\; by = -ax + c \;\Rightarrow\; y = -\frac{a}{b}x + \frac{c}{b},$$

so here the gradient of the line is equal to $-\frac{a}{b}$, and the y-intercept is equal to $\frac{c}{b}$. We will use these equations to define systems of linear equations.

Systems of linear equations

A **system of linear equations** is a collection of two or more linear equations involving the same variables. In the simplest case, we have two equations in two variables:

$$\begin{cases} ax + by = c, \\ dx + ey = f, \end{cases}$$

where a, b, c, d, e, f are numbers and x, y are variables.

For this lecture we will only focus on the case of systems of two equations in two variables – we will reserve the general theory for next lecture.

1.2 Solutions of Systems of Equations

In our language analogy, an equation is making a claim that relates two algebraic expressions together – it is like saying "this *is* that". The solutions to an equation are simply all of the values of the variables that make a true statement after substituting them in to the equation. In this sense, we can understand a system of two linear equations as behaving much like an "and" connective – it is like saying "this is that *and* this is that". The only way that an "and" statement like this would be true is if both equations are true at the same time. This brings us to the definition of a solution of a system of linear equations.

Solutions of a system

A **solution** to a system of equations in x and y is an ordered pair (x, y) that satisfies **all** equations in the system **at the same time**.

As an example, consider the system of linear equations:

$$\begin{cases} x + y = 6, \\ 2x - 5y = -2. \end{cases}$$

We will now check whether or not the points $(3, 3)$ and $(4, 2)$ are solutions.

- $(3, 3)$: we substitute $x = 3$ and $y = 3$ into both equations. For the first equation we have

  $$(3) + (3) = 6 \ \Rightarrow 6 = 6, \qquad \text{therefore } (3,3) \text{ is a solution to } x + y = 6.$$

  However, when we substitute $(3, 3)$ into the second equation we get:

  $$2(3) - 5(3) = -2 \ \Rightarrow 6 - 15 \neq -2, \qquad \text{therefore } (3,3) \text{ is not a solution to } x + y = 6.$$

  Since the point $(3, 3)$ is *not* a solution to the second equation, this is not a solution to the system either.
- $(4, 2)$: we substitute $x = 4$ and $y = 2$ into both equations. Now we see that

  $$(4) + (2) = 6 \ \Rightarrow 6 = 6, \qquad \text{therefore } (4,2) \text{ is a solution to } x + y = 6,$$

  and also:

  $$2(4) - 5(2) = -2 \ \Rightarrow 8 - 10 = -2, \qquad \text{therefore } (3,3) \text{ is a solution to } x + y = 6.$$

  Since the pair $(4, 2)$ is a solution to both equations at the same time, it is also a solution to the system.

Common mistake

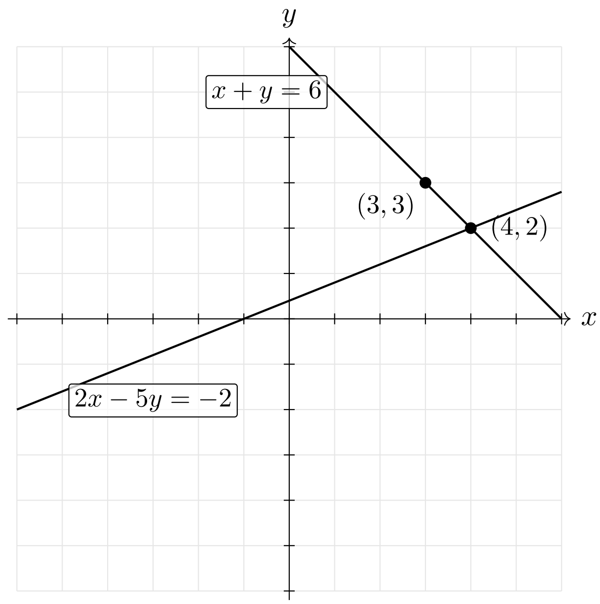
A solution to a system of equations must be a solution to *every* equation at the same time. As the previous example demonstrates, it is not enough to simply check that a proposed solution works for one of the equations.

2 Solving a System by Graphing

A linear equation in x and y describes the geometry of a line. So, when we have a system of two linear equations in x and y, we have geometric descriptions of **two lines** at the same time. Consider again the system:

$$\begin{cases} x + y = 6, \\ 2x - 5y = -2. \end{cases}$$

Here, we can plot the two equations $x + y = 6$ and $2x - 5y = -2$ on the same axes:



Notice the two labelled points are $(3, 3)$ and $(4, 2)$ from the previous example. In the case of $(3, 3)$, the point lies on the first line, since it was a solution to the equation $x + y = 6$. However, it the point does *not* lie on the second line, since it was not a solution to the equation $2x - 5y = -2$. Meanwhile, the point $(4, 2)$, which was the solution to our system, lies on both lines at the same time. This is a general feature of solutions to systems of linear equations.

Graphical meaning of a solution

A point (x, y) solves the system

$$\begin{cases} ax + by = c, \\ dx + ey = f \end{cases}$$

if and only if (x, y) lies on **both** lines. So, the solution(s) are exactly the **intersection point(s)** of the two graphs.

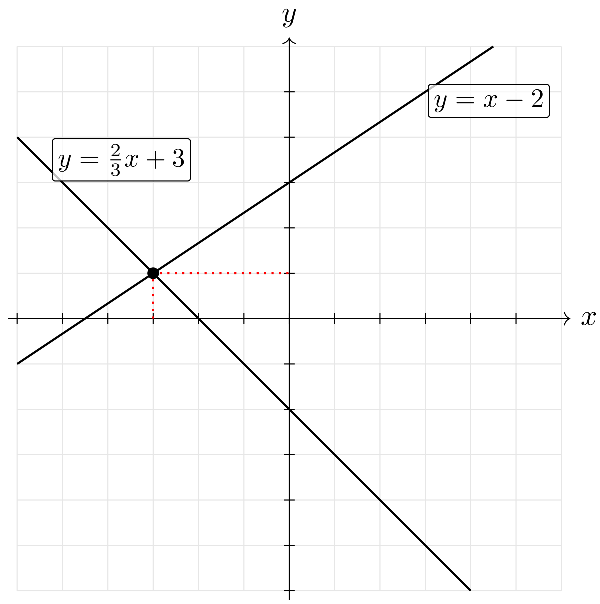
As another example, consider the system:

$$\begin{cases} x + y = -2, \\ 2x - 3y = -9. \end{cases}$$

In order to solve this by graphing the lines, it is easiest to first write the equations in gradient-intercept form. From $x + y = -2$ we get $y = -x - 2$, and from $2x - 3y = -9$ we get

$$-3y = -2x - 9 \quad \Rightarrow \quad y = \frac{2}{3}x + 3.$$

The diagram below depicts both graphs.



Observe that the lines intersect at the point $(-3, 1)$ – this is the solution to our system of linear equations. We can check that this is the case by substituting these values into both equations and verifying that the resulting statements are true:

$$(-3) + (1) = -2 \qquad \text{and} \qquad 2(-3) - 3(1) = -9.$$

2.1 Three possible outcomes

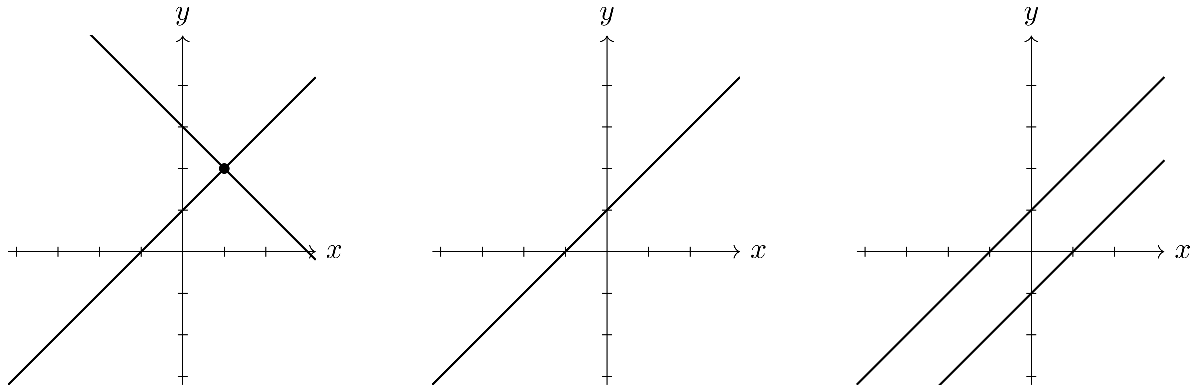
According to the above, if we can plot the lines described by a system of linear equations, then we can often determine solutions to the system by simply inspecting where the two lines cross. However, it is not always the case that lines cross each other at a single point, which would imply that certain systems may not have a unique solution. As a matter of fact, there are three possible ways that a pair of lines can interact.

Consistent, inconsistent, and dependent systems

Consider a system of two linear equations in two variables.

1. **One solution (consistent):** the lines intersect once (different slopes).
2. **No solution (inconsistent):** the lines are parallel (same slope, different intercept).
3. **Infinitely many solutions (dependent):** the lines are the same line (the equations are equivalent).

The three types of interacting lines are pictured below.



One solution **Infinitely many solutions** **No solutions**

In the first case, the two lines have a single point of intersection. In the second case, the two lines intersect *everywhere*, and therefore there would be infinitely-many solutions corresponding to that system. In the third case, the lines are parallel and therefore *never* meet, meaning that the associated system would not have any solutions at all.

Exercise 2

Use a graph (or your intuition about slopes) to decide how many solutions each system has.

1. $\begin{cases} x + y = -2 \\ 2x - 3y = -9 \end{cases}$
2. $\begin{cases} y = 2x + 1 \\ y = 2x - 3 \end{cases}$
3. $\begin{cases} 2y = 3x + 2 \\ 4y = 6x + 4 \end{cases}$
4. $\begin{cases} x - 3y = 2 \\ -2x + 6y = 2 \end{cases}$

3 Solving a System by Substitution

Although geometrically-pleasing, the method of graphing lines is not always the best approach to solving a system of linear equations. The simplest reason would be that in reality, perhaps the solution of the system is not a nice pair of integers, and therefore the graph does not perfectly tell us what the solution is. It would be better to use algebra instead, so that we can obtain precise descriptions of the solutions of a system, no matter what the graphs looked like.

The method of substitution is one such algebraic approach to solving systems of two linear equations. It is perhaps the most intuitive algebraic approach.

The method of substitution

To solve a system by **substitution**:

1. Solve one equation for one variable (for example, solve for y).
2. Substitute that expression into the other equation to get an equation in one variable.
3. Solve the resulting one-variable equation.
4. Substitute this answer back into the other equation to find the other variable.
5. Check the solution in the original equations.

The step-by-step process may seem a little unusual at first, but it becomes very clear once we see an example. Consider the system of equations:

$$\begin{cases} -x + y = 1, \\ 2x + y = -2. \end{cases}$$

The first goal of our method is to somehow convert one of the equations into a usable form that can be substituted into other. That way, we reduce our algebra from equations involving two variables into a single equation involving one variable. From the first equation, we can rearrange to get

$$-x + y = 1 \quad \Rightarrow \quad y = x + 1.$$

Now that we have an expression of y purely in terms of x, we can substitute it into the second equation:

$$2x + y = -2 \quad \Rightarrow \quad 2x + (x+1) = -2 \quad \Rightarrow \quad 3x + 1 = -2 \quad \Rightarrow \quad 3x = -3 \quad \Rightarrow \quad x = -1.$$

This gives us one half of the total solution. To find the value of y, we can substitute $x = -1$ into one of the equations and solve for y. It is easiest to use the first equation, i.e. we substitute $x = -1$ into $y = x + 1$:

$$y = -1 + 1 = 0.$$

So the solution is $(x, y) = (-1, 0)$.

3.1 An example with fractions

Consider the system:

$$\begin{cases} 5x + 3y = 18, \\ 2x - 7y = -1. \end{cases}$$

We will solve this via substitution. We start by rearranging the first equation in order to solve for x:

$$5x + 3y = 18 \quad \Rightarrow \quad 5x = -3y + 18 \quad \Rightarrow \quad x = -\frac{3}{5}y + \frac{18}{5}.$$

We now substitute this unfortunate-looking value of x into our second equation:

$$2x - 7y = -1 \quad \Rightarrow \quad 2\left(-\frac{3}{5}y + \frac{18}{5}\right) - 7y = -1.$$

We then solve this for y. First, we can simplify:

$$-\frac{6}{5}y + \frac{36}{5} - 7y = -1.$$

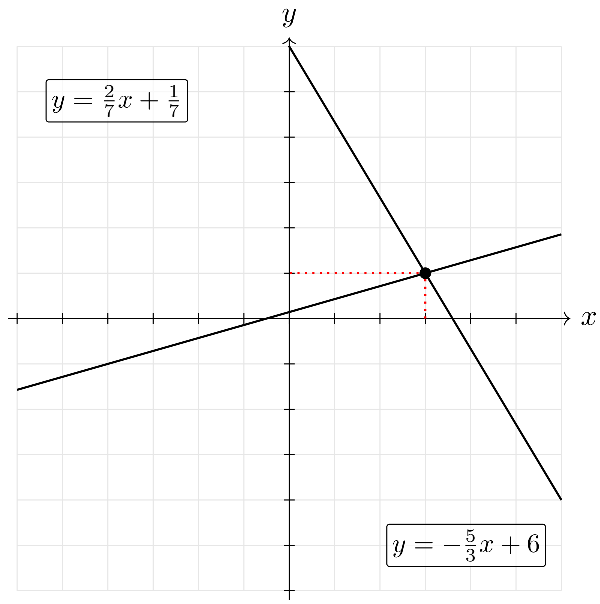
Now we can multiply both sides by 5 and rearrange the equation:

$$-6y + 36 - 35y = -5 \quad \Rightarrow \quad 36 - 41y = -5 \quad \Rightarrow \quad -41y = -41 \quad \Rightarrow \quad y = 1.$$

This is one half of our final answer. To find the x value of the solution, we substitute the value of y back into the rearranged equation $x = -\frac{3}{5}y + \frac{18}{5}$:

$$x = -\frac{3}{5}(1) + \frac{18}{5} = \frac{15}{5} = 3.$$

So the solution is $(3, 1)$. This can be confirmed by graphing the equations:



3.2 Substitution with dependent or inconsistent systems

Sometimes substitution produces a surprising statement. For example, if we try to solve the system

$$\begin{cases} x - 3y = 2 \\ -2x + 6y = 2 \end{cases}$$

using substitution, we will see that $x = 3y + 2$ gives

$$-2(3y + 2) + 6y = 2 \quad \Rightarrow \quad -6y + 4 + 6y = 2 \quad \Rightarrow \quad 4 = 2.$$

This means that no matter which value of y we choose, we will always obtain a contradiction. Therefore, there is *no* solution to this system, i.e. the system is inconsistent.

As another example, consider now the system

$$\begin{cases} 9x + 3y = 15 \\ 3x + y = 5 \end{cases}$$

Here, if we try to use substitution, we see that the first equation gives $y = -3x + 5$, which gives the substitution:

$$9x + 3(-3x + 5) = 15 \quad \Rightarrow \quad 9x - 9x + 15 = 15 \quad \Rightarrow 15 = 15.$$

This equation is true no matter what value we choose for x. Therefore, this system does not just have one solution, but it has *infinitely many* corresponding to the different values of x. In other words, this is a dependent system.

Interpreting the final statement

After substitution and simplification:

1. If you get a false statement (for example $-4 = 2$), the system is **inconsistent** and has **no solution**.
2. If you get a true statement (for example $15 = 15$), the system is **dependent** and has **infinitely many solutions** (the two equations describe the same line).

Exercise 3

Solve each system by substitution.

1. $\begin{cases} -x + y = 1 \\ 2x + y = -2 \end{cases}$

2. $\begin{cases} 9x + 3y = 15 \\ 3x + y = 5 \end{cases}$

4 Solving a System by Elimination

We now move on to our final method for solving systems of two linear equations: Elimination. Like substitution, this is an algebraic process that may give exact results. However, it is a different technique that will be very useful when trying to solve more complicated systems in the next lecture. The method can be summarised as follows.

The method of elimination

To solve a system by **elimination**:

1. Obtain opposite coefficients of x (or y) by multiplying one or both equations by suitable constants.
2. Add the equations to eliminate one variable and solve the resulting one-variable equation.
3. Back-substitute into one of the original equations to find the other variable.
4. Check your solution in both original equations.

Put differently, the method of elimination seeks to set up the pair of equations in such a way that the coefficients of one of the variables line up with opposite sign. Then we can add the two equations together and it will "eliminate" this variable, leaving us with an equation in one variable that is easy to solve.

4.1 Two examples

As a simple example of the method of elimination, consider the system:

$$\begin{cases} 4x + 3y = 1, \\ 2x - 3y = 5. \end{cases}$$

Notice that the two y terms appear with the same coefficients with opposite sign (3 and -3). So, we can add the two equations together to eliminate y:

$$(4x + 3y) + (2x - 3y) = 1 + 5 \quad \Rightarrow \quad 6x = 6 \quad \Rightarrow \quad x = 1.$$

We can now substitute this value of x back into the first equation and solve for y:

$$4(1) + 3y = 1 \quad \Rightarrow \quad 3y = -3 \quad \Rightarrow \quad y = -1.$$

So the solution is $(1, -1)$.

Notice that in this example, we started with a system in which a variable existed with the same coefficient and opposite sign. From there, it was easy to eliminate this variable by adding the equations together. However, in general it may not be the case that coefficients in the system match up immediately. So, an important step in the method of elimination is to first arrange the equations in such a way that one of the variables has matching coefficients.

To see an example of this, consider the system:

$$\begin{cases} 2x - 3y = -7, \\ \;\; 3x + y = -5. \end{cases}$$

Notice that here, neither of the variables x or y has their coefficients matching. But, we can manipulate the equations in order to guarantee this. Since it will be easier to do, we will modify the y coefficients, which are currently -3 and 1. If we multiply the second equation by 3 on both sides, then we will not change the underlying geometry of the line described by the equation $3x + y = -5$. This gives us a *new* system of equations for which the coefficients of y now match up:

$$\begin{cases} 2x - 3y = -7, \\ \;\; 3x + y = -5. \end{cases}$$

From here, we may proceed as before and add these two equations together to eliminate y:

$$(2x - 3y) + (9x + 3y) = -7 + (-15) \quad \Rightarrow \quad 11x = -22 \quad \Rightarrow \quad x = -2.$$

We then take this value of x and substitiute it into one of the original equations to solve for y. We substitute into $3x + y = -5$ to get:

$$3(-2) + y = -5 \quad \Rightarrow \quad -6 + y = -5 \quad \Rightarrow \quad y = 1.$$

So, the solution is $(-2, 1)$.

4.2 Elimination with inconsistent or dependent systems

As with substitution, elimination can also produce unexpected outcomes. We will now try to solve the two systems of Section 3.2 using elimination.

Consider first the system

$$\begin{cases} x - 3y = 2, \\ -2x + 6y = 2. \end{cases}$$

We can match up the two If we multiply the first equation by 2, we will obtain the new system:

$$\begin{cases} 2x - 6y = 4, \\ -2x + 6y = 2. \end{cases}$$

Now, when we add these two equations to eliminate x, we will get:

$$-6y + 6y = 4 + 2 \quad \Rightarrow \quad 0 = 6.$$

In the process of trying to eliminate x, we also *accidentally* eliminate y and obtain the false statement $0 = 6$. This mirrors the conclusion we made in Section 3.2. – this system does not have a solution.

Consider now the system

$$\begin{cases} 9x + 3y = 15 \\ 3x + y = 5 \end{cases}$$

In this case, we can multiply the second row by 3 to prepare the coefficients of x for elimination. We obtain the new system:

$$\begin{cases} 9x + 3y = 15 \\ 9x + 3y = 15 \end{cases}$$

Now, we can try to eliminate the x variable by subtracting the second equation from the first. However, we happen to have two copies of the *same* equation, so when subtracting one from the other everything will cancel on both sides and leave us with the identity $0 = 0$. Of course, the statement $0 = 0$ is always true no matter what, and therefore we conclude that this system is dependent and has infinitely many solutions.

These two phenomena are summarized below.

0

After elimination:

1. If you get $0 = k$ where $k \neq 0$ (for example $0 = 11$), the system is **inconsistent** and has **no solution** (parallel lines).
2. If you get $0 = 0$, the system is **dependent** and has **infinitely many solutions** (the same line).

There is also a geometric narrative connecting these conclusions to that Section 3.2. Two different-looking equations can describe the **same** line. This happens precisely when one equation is a nonzero constant multiple of the other (after moving everything into standard form).

Same line means scalar multiples

If two linear equations describe the same line, then (after writing both in standard form) one equation is a nonzero constant multiple of the other. For example:

$$2y = 3x + 2 \quad \text{and} \quad 4y = 6x + 4$$

describe the same line, because the second equation is 2 times the first.

Parallel lines have the same slope but different intercepts. For example, $y = mx + b_1$ and $y = mx + b_2$ with $b_1 \neq b_2$ are parallel and never intersect.

Exercise 4

Solve each system by elimination, or decide that there is no unique solution.

1. $\begin{cases} 2b + a = 15 \\ 3b - a = 5 \end{cases}$
2. $\begin{cases} 4x + 3y = 1 \\ 2x - 3y = 5 \end{cases}$
3. $\begin{cases} 2x - 3y = -7 \\ 3x + y = -5 \end{cases}$
4. $\begin{cases} 2x - 6y = 5 \\ 3x - 9y = 2 \end{cases}$
5. $\begin{cases} 2x - 6y = -5 \\ -4x + 12y = 10 \end{cases}$

Solutions to the Exercises

Exercise 1

1. $(3, 3)$: Equation 1 gives $3 + 3 = 6$ (true). Equation 2 gives $2(3) - 5(3) = 6 - 15 = -9 \neq -2$ (false). So $(3, 3)$ is not a solution.
2. $(4, 2)$: Equation 1 gives $4 + 2 = 6$ (true). Equation 2 gives $2(4) - 5(2) = 8 - 10 = -2$ (true). So $(4, 2)$ is a solution.
3. $(1, 5)$: Equation 1 gives $1 + 5 = 6$ (true). Equation 2 gives $2(1) - 5(5) = 2 - 25 = -23 \neq -2$ (false). So $(1, 5)$ is not a solution.

Exercise 2

1. One solution (the lines intersect once), at $(-3, 1)$.
2. No solution (parallel lines: same slope 2, different intercepts).
3. Infinitely many solutions (same line: the second equation is 2 times the first).

4. No solution (inconsistent: multiplying $x-3y=2$ by -2 gives $-2x+6y=-4$, which contradicts $-2x+6y=2$).

Exercise 3

1. $y=x+1$ from $-x+y=1$. Substitute into $2x+y=-2$: $2x+(x+1)=-2 \Rightarrow 3x=-3 \Rightarrow x=-1$, then $y=0$. So $(-1,0)$.
2. From the worked example, the solution is $(3,1)$.
3. From $x-3y=2$ we get $x=3y+2$. Substitute into $-2x+6y=2$: $-2(3y+2)+6y=2 \Rightarrow -6y-4+6y=2 \Rightarrow -4=2$ (false). No solution.
4. The second equation times 3 is the first, so the lines coincide. Infinitely many solutions: all (x,y) satisfying $3x+y=5$.

Exercise 4

1. Add: $(2b+a)+(3b-a)=15+5 \Rightarrow 5b=20 \Rightarrow b=4$, then $a=15-2(4)=7$.
2. Add: $(4x+3y)+(2x-3y)=1+5 \Rightarrow 6x=6 \Rightarrow x=1$, then $4(1)+3y=1 \Rightarrow y=-1$. So $(1,-1)$.
3. Multiply the second equation by 3: $9x+3y=-15$. Add to $2x-3y=-7$: $11x=-22 \Rightarrow x=-2$, then $3(-2)+y=-5 \Rightarrow y=1$. So $(-2,1)$.
4. Multiply $2x-6y=5$ by 3 to get $6x-18y=15$. Multiply $3x-9y=2$ by -2 to get $-6x+18y=-4$. Add: $0=11$ (false). No solution.
5. Multiply $2x-6y=-5$ by 2 to get $4x-12y=-10$. Add to $-4x+12y=10$ to get $0=0$ (true). Infinitely many solutions, namely all (x,y) on the line $2x-6y=-5$.