MAT140 - Lecture 14 Handout

Systems of Linear Equations

Last lecture we solved systems of two linear equations using either graphing, substitution or elimination. We saw that graphing was the most beautiful, substitution was the most intuitive, and elimination was the most useful. Regarding the latter, we briefly mentioned that the process of elimination is the best method of the three because it scales to more complicated systems in a way that substitution and graphing cannot. Today we will flesh out this idea by seeing the full power of elimination.

We will now study more complicated systems of linear equations in which there are *three or more* equations, which have extra unknowns. Our strategy for solving these systems will be to first convert them into a form that is easier to solve, and then to solve this easier thing. This strategy has a series of technical names attached to it, and the entire process is called *Gaussian Elimination* – very similar, in fact, to the technique of elimination we saw in the previous lecture. In technical terms, today we will:

1. Understand row-echelon form and solve simple systems using back-substitution.
2. Learn the three row operations and see how they preserve the solution set of a system.
3. Use Gaussian elimination to solve a system of linear equations.
4. Introduce matrices – a notation that simplifies the process of Gaussian Elimination.

In what follows, we will focus mainly on the setting of systems of linear equations made of either 2 or 3 equations/unknowns. But, it should be noted that the techniques we will discuss will also apply to more complicated systems.

1 Row-Echelon Form and Back-Substitution

When we use Gaussian Elimination, the goal is to rewrite the system into a form where the last equation has only one variable, the second-last has only two variables, and so on. In a sense, we can see that we "eliminate" variables. This produces a triangular, "step" pattern.

Row-echelon form

A system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

is in **row-echelon form** if the coefficients a_2, a_3 and b_3 are all equal to zero, so that:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ \qquad\quad b_2y + c_2z = d_2 \\ \qquad\qquad\quad c_3z = d_3. \end{cases}$$

For example, compare the following two systems:

$$\begin{cases} x - 2y + 2z = 9, \\ -x + 3y = -4, \\ 2x - 5y + z = 10. \end{cases} \qquad \begin{cases} x - 2y + 2z = 9, \\ y + 2z = 5, \\ z = 3. \end{cases}$$

Not row-echelon form **Row-echelon form**

Conveniently, the latter can be solved using an intuitive process known as *back-substitution*. If we have a system written in row-echelon form, like the right-hand system above, then we can work our way from the bottom to the top by solving the system one variable at a time. For example, the system:

$$\begin{cases} x - 2y + 2z = 9, \\ y + 2z = 5, \\ z = 3. \end{cases}$$

immediately presents the solution for the value of z – the third equation simply tells us that $z = 3$. Now that we know this, we can substitute this value of z into the second equation and solve for y:

$$y + 2(3) = 5 \quad \Rightarrow \quad y + 6 = 5 \quad \Rightarrow \quad y = -1.$$

Now that we know that both $z = 3$ and $y = -1$, we can substitute these two values into the first equation and solve for x:

$$x - 2(-1) + 2(3) = 9 \quad \Rightarrow \quad x + 2 + 6 = 9 \quad \Rightarrow \quad x = 1.$$

Therefore the solution is $(1, -1, 3)$.

Exercise 1

Solve the following system using back-substitution:

$$\begin{cases} 4a + b + c = 9, \\ 3b - 2c = 5, \\ 3c = 6. \end{cases}$$

2 Gaussian Elimination and Row Operations

Gaussian elimination is the systematic process of converting a system into row-echelon form, without changing its set of solutions. If we can do this, then it's possible to solve a system using the simple process of back-substitution, without needing to solve any complicated equations involving multiple unknowns. We will now spend some time discussing *how* we can convert a system into row-echelon without accidentally spoiling its solution space.

2.1 Deriving the Row Operations

Ultimately, we are looking for operations which may help us to simplify a system *without changing its solutions.* We will now present a few ways to change a system's equations without changing its solutions.

Method 1: Interchanging Rows. To begin with, it should be obvious that switching the order in which we state the equations does nothing to the geometry underlying them.[1]

Method 2: Rescaling by a constant. Another way to change the equation of a line without changing the line itself is to multiply both sides by the same nonzero constant. This does not move the points of the line, it simply gives a different equation for the same line.

Method 3: Adding the two equations together. Finally, we have a rather subtle observation: the solution space of a pair of equations will be preserved whenever we *add* the two equations together. To see this algebraically, let's write out a general system of 2 equations and 2 unknowns:

$$\begin{cases} a_1x + b_1y = c_1, & (1) \\ a_2x + b_2y = c_2. & (2) \end{cases}$$

Suppose that we know the point (α, β) is a solution to this system. This means that when we substitute $x = \alpha$ and $y = \beta$ into equation (1) or (2), then we get a true statement. Adding these two equations together means to add the left-hand-sides and add the right-hand-sides together, and then set these two expressions equal to each other. Writing this out, we have:

$$(a_1x + b_1y) + (a_2x + b_2y) = c_1 + c_2 \quad \Rightarrow \quad (a_1 + a_2)x + (b_1 + b_2)y = c_1 + c_2.$$

Now, we notice that this new equation is *still* linear, since we have not added any extra powers to either x or y.[2] Moreover, this new linear equation *still* has $x = \alpha$ and $y = \beta$ as solutions. If we substitute these values in, we would get:

$$\underbrace{(a_1(\alpha) + b_1(\beta))}_{\text{equals } c_1} + \underbrace{(a_2(\alpha) + b_2(\beta))}_{\text{equals } c_2} = c_1 + c_2.$$

Geometrically, this new linear equation will generally describe a completely different line from the first two in our system. However, according to the algebra above, this new line will still intersect the graphs of Equations (1) and (2) at the point (α, β). Therefore, although we have changed the equation of a line, we have **not** changed the solution itself.

To see this in action, let's consider the system of equations

$$\begin{cases} 4x + 4y = 16, & (1) \\ -2x + 2y = 4. & (2) \end{cases}$$

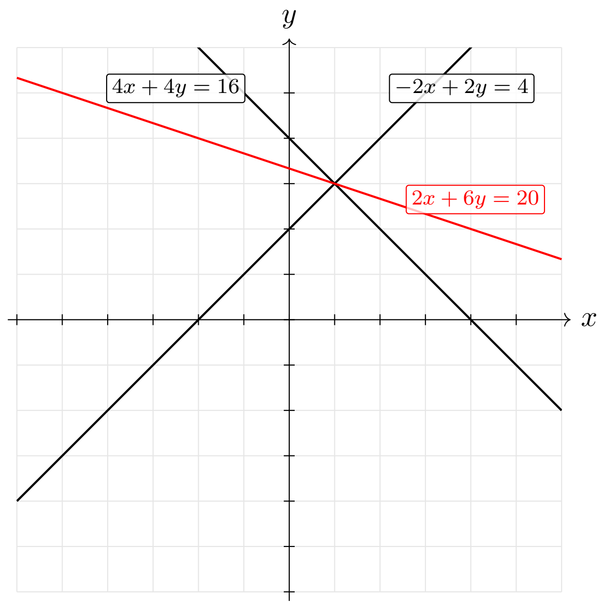
[1] A system of two equations is like saying "this *and* that must be true at the same time". But, the word "and" is symmetric – saying A and B is the same as saying B and A.

[2] It may help to notice that the new equation is still first-degree in x and y, so it is still a linear equation.

You may verify for yourself that this system has the solution $x = 1$ and $y = 3$. Graphically, this means that these two lines will intersect at the coordinate point $(1, 3)$. If we take the sum of equation (1) and (2), we obtain the equation of a new line:

$$(4x + 4y) + (-2x + 2y) = 16 + 4 \quad \Rightarrow \quad 2x + 6y = 20.$$

This new equation describes a line that is totally different from the first two. However, notice what happens when we plot the three lines together:



As you can see, the sum of the two lines defines a third line (drawn in red) which *still* intersects the other two at $(1, 3)$. This means that we could effectively replace one of the original with this third line and we would not be changing the solution of the system. In other words:

$$\begin{cases} 4x + 4y = 16, & (1) \\ -2x + 2y = 4, & (2) \end{cases} \quad \text{is equivalent to} \quad \begin{cases} 4x + 4y = 16, & (1) \\ 2x + 6y = 20. & (2) \end{cases}$$

In the above, we have replaced the equation (2) with the sum of (1) and (2).

2.2 The Three Row Operations

Generally speaking, we can perform the three methods detailed above in order to transform a system of equations whilst preserving the solutions.

The three row operations

There are three basic **row operations**:

(O1) Swap rows: switch the order of two equations.

(O2) Scale a row: multiply an equation by a nonzero constant.

(O3) Replace a row: add a multiple of one equation to another equation.

Although it is slightly beyond the scope of the course: it is worth mentioning that these three operations are in some sense all that there is – it can be shown that you only need three such operations, and that *every* other operation that preserves the solution set of a system of linear equations can be rewritten as a combination of the above three operations.

We will now present a brief example of the three row operations. Consider the system

$$\begin{cases} x - 2y + 2z = 9 \\ -x + 3y = -4 \\ 2x - 5y + z = 10 \end{cases}$$

As an example of **(O1)**, we will interchange Row 1 and Row 2. This is written:

$$\begin{cases} x - 2y + 2z = 9 \\ -x + 3y = -4 \\ 2x - 5y + z = 10 \end{cases} \xrightarrow{R_1 \leftrightarrow R_2} \begin{cases} -x + 3y = -4 \\ x - 2y + 2z = 9 \\ 2x - 5y + z = 10 \end{cases}$$

As an example of (O2), we multiply the first row by 3:

$$\begin{cases} x - 2y + 2z = 9 \\ -x + 3y = -4 \\ 2x - 5y + z = 10 \end{cases} \xrightarrow{R_1 \to 3R_1} \begin{cases} 3x - 6y + 6z = 27 \\ -x + 3y = -4 \\ 2x - 5y + z = 10 \end{cases}$$

As an example of (O3), we will replace Row 1 with the sum of Rows 1 and 2:

$$\begin{cases} x - 2y + 2z = 9 \\ -x + 3y = -4 \\ 2x - 5y + z = 10 \end{cases} \xrightarrow{R_1 \to R_1 + R_2} \begin{cases} y + 2z = 5 \\ -x + 3y = -4 \\ 2x - 5y + z = 10 \end{cases}$$

2.3 A Worked example of Gaussian Elimination

As an example of the row operations in action, let's consider the system:

$$\begin{cases} x - 2y + 2z = 9, \\ -x + 3y + 0z = -4, \\ 2x - 5y + z = 10. \end{cases}$$

We are going to solve this system by reducing it to something written in row echelon form, so that we can easily solve it using back-substitution. In order to do this, we start by eliminating the two x terms that are in the second and third rows. In order to do this, we need to first inspect what coefficients the x terms have, and identify the relationship between this number and the x term in the first row. The critical question that we should then answer is:

> "What number do I have to multiply the first row by so that it will cancel the x term in the other row once we add the equations together?"

For the elimination of the x terms, we have to answer this question twice, and then finally we need to also eliminate the y term in Row 3. We do these three steps in order:

- For Row 2, we notice that the x term is $-x$ whereas the x term in Row 1 is simply x. Therefore, if we add these two rows together, the x's will cancel each other out and therefore be eliminated. We want to put this "x-eliminated row" into Row 2 so that there is no longer an x term there. So, the correct row operation to use is an application of (O3): $R_2 \to R_2 + R_1$.

$$(-x+3y)+(x-2y+2z) = -4+9 \quad \Rightarrow \quad y+2z=5.$$

  Replacing Row 2 with this answer gives the new system:

$$\begin{cases} x-2y+2z=9, \\ \qquad\; y+2z=5, \\ 2x-5y+z=10. \end{cases}$$

  and we see that we have successfully eliminated the x term in the second row.

- For Row 3, we notice that the x term is $2x$, whereas Row 1 has an entry of simply x. The critical question is then: what number should we multiply Row 1 by so that it's new coefficient matches $2x$ and will therefore cancel when we add the equations? The answer here is that we want to somehow convert the x term in Row 1 into a $-2x$, so that we get $2x-2x=0$ when we eventually sum. Therefore, our row operation will be another application of (O3), this time we replace Row 3 by the sum of Row 3 and -2 times Row 1. In symbols, we do: $R_3 \to R_3 - 2R_1$. Written out, this operation is:

$$(2x-5y+z)-2(x-2y+2z) = 10-18 \quad \Rightarrow \quad -y-3z=-8.$$

  We replace Row 3 with this answer, so that our new system is:

$$\begin{cases} x-2y+2z=9, \\ \qquad\; y+2z=5, \\ \quad\; -y-3z=-8. \end{cases}$$

- Finally, we need to eliminate the y term from Row 3. To do this, we notice that if we try to add/subtract some copy of Row 1 to Row 3, then we will unintentionally introduce an x term in the answer and therefore we will undo the work we have just done. So, to eliminate the y term in Row 3 we need to use Row 2 to do it. We start by inspecting the coefficients of the y's in these two rows and comparing them. Notice that in our system above, the y's in Row 2 and 3 already have the same coefficient (namely 1), and that they are simply the same but with opposite sign. Therefore, we can eliminate the y term in Row 3 by adding Row 2 and 3 together. In symbols, we need to do the (O3) operation: $R_3 \to R_3 + R_2$. Adding these two eliminates the y term in Row 3:

$$(-y-3z)+(y+2z) = -8+5 \quad \Rightarrow \quad -z=-3.$$

  and therefore we are left with the row-echelon form:

$$\begin{cases} x-2y+2z=9, \\ \qquad\; y+2z=5, \\ \qquad\quad -z=-3. \end{cases}$$

This is the same system that we discussed in Section 1, and it can be solved by back-substitution to yield the solution $x=1$, $y=-1$ and $z=3$.

2.4 An inconsistent system

Sometimes Gaussian elimination produces a false statement, which means that the system has no solution. This is completely the same concept as the inconsistent systems we saw for systems of two equations – we perform the correct logical steps to arrive at a solution, but then arrive at a statement that is always false.

As an example of this, consider the system:

$$\begin{cases} x - 3y + z = 1, \\ 2x - y - 2z = 2, \\ x + 2y - 3z = -1. \end{cases}$$

A sequence of row operations can produce the equivalent system:

$$\begin{cases} x - 3y + z = 1, \\ \quad 5y - 4z = 0, \\ \quad 5y - 4z = -2. \end{cases}$$

Subtracting the second equation from the third gives $0 = -2$, which is false. Alternatively, we may notice that within this system of three equations, there is a *subsystem* of two equations with two unknowns that is inconsistent:

$$\begin{cases} x - 3y + z = 1, \\ \boxed{\begin{aligned} 5y - 4z &= 0, \\ 5y - 4z &= -2. \end{aligned}} \longleftarrow \end{cases}$$

Graphically, these two equations would correspond to two parallel lines in the yz-plane.

Generally speaking, for all systems of linear equations there are only a small number of possibilities.

Number of solutions (Gaussian elimination outcome)

For a system of linear equations, exactly one of the following is true:

1. There is exactly one solution.
2. There are infinitely many solutions.
3. There is no solution.

3 Matrices and Augmented Matrices

In a sense, when we perform row operations to solve a system of linear equations, we don't fully care about the variables, but instead only their *coefficients*. When looking to cancel variables, we inspect the coefficients in two different rows and try to figure out how to cancel these coefficients to a zero using (O3). As a matter of fact, we could get away with simply writing the coefficients and the row operations, without really caring about the letters themselves. Consider a simple version of this:

$$\begin{cases} 4x + y = 2 \\ 2x + 2y = 1 \end{cases} \xrightarrow[\color{red}{(O2)}]{(R2) \to 2\cdot(R2)} \begin{cases} 4x + y = 2 \\ 4x + 4y = 2 \end{cases} \xrightarrow[\color{red}{(O3)}]{(R2) \to (R2) - (R1)} \begin{cases} 4x + y = 2 \\ 3y = 0 \end{cases}$$

$$\Big\downarrow \text{ Write just the numbers} \qquad \Big\downarrow \text{ Write just the numbers} \qquad \Big\downarrow \text{ Write just the numbers}$$

$$\left[\begin{array}{cc|c} 4 & 1 & 2 \\ 2 & 2 & 1 \end{array}\right] \xrightarrow[\color{red}{(O2)}]{(R2) \to 2\cdot(R2)} \left[\begin{array}{cc|c} 4 & 1 & 2 \\ 4 & 4 & 2 \end{array}\right] \xrightarrow[\color{red}{(O3)}]{(R2) \to (R2) - (R1)} \left[\begin{array}{cc|c} 4 & 1 & 2 \\ 0 & 3 & 0 \end{array}\right]$$

In a way, the collection of numbers arranged in a grid displays all of the same information in a simpler notation. Importantly,

- we can still see the arrangement of rows,
- we can still see the equals sign (here represented by the vertical bar),
- we can still identify the coefficients corresponding to each variable, and
- we can still perform the row operations and construct a row-echelon form for our system.

This means that if we simply write out the coefficients of the terms involved in our system, then we don't actually lose any important information. So, we can still solve our system, but now in a streamlined notation.

These gridlike arrays of numbers are very useful in practice. They have a name: *matrices*.

Matrices and their order

A **matrix** is a rectangular array of numbers. We write a matrix with m rows and n columns as:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

If it has m rows and n columns, we say that its **order** is $m \times n$.

Exercise 3

Determine the order of each matrix:

(a) $\begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -3 \\ -2 & 0 \\ 4 & -2 \end{pmatrix}$

3.1 Coefficient and augmented matrices

A system of linear equations (with constant terms on the right) has a **coefficient matrix** and an **augmented matrix**.

Coefficient matrix and augmented matrix of a system of 3 equations

Consider a system of three equations and three unknowns:

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

The **coefficient matrix** is the 3×3 matrix formed by taking all of the coefficients on the left hand side of the equation and arranging them into the matrix:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

The **augmented matrix** of the above system adds an extra column to the coefficient matrix, so that the constants on the right-hand side are included:

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right].$$

We will write augmented matrices with square brackets to distinguish them from ordinary matrices.

As an example, consider the system:

$$\begin{cases} 3x + 2y - z = 1, \\ x + 0y + 2z = -3, \\ -2x - y + 0z = 4. \end{cases}$$

The numbers multiplying the variables can be placed into a matrix:

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & 0 & 2 \\ -2 & -1 & 0 \end{pmatrix}.$$

This is called the *coefficient matrix*. If we include the constants on the right-hand side as an extra column, we obtain the *augmented matrix*:

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 1 & 0 & 2 & -3 \\ -2 & -1 & 0 & 4 \end{array}\right].$$

Although we just defined the notion of augmented matrices for a system of three equations, it should be noted that there is no restriction here – we can form augmented matrices for any system of linear equations. For example, the augmented matrices of the systems at the start of Section 3 are 2×3 matrices.

Exercise 4

Form the coefficient matrix and augmented matrix for each system.

1. $\begin{cases} -x + 5y = 2 \\ 7x - 2y = -6 \end{cases}$

2. $\begin{cases} 3x + 2y - z = 1 \\ x + 0y + 2z = -3 \\ -2x - y + 0z = 4 \end{cases}$

4 Elementary Row Operations on Matrices

We can apply the same row operations directly to augmented matrices.

Exercise 5

Perform each row operation.

1. Interchange the first and second rows:

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{array}\right]$$

2. Multiply the first row by $\frac{1}{2}$:

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{array}\right]$$

3. Add -2 times the first row to the third row:

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{array}\right]$$

4. Add 6 times the first row to the second row:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ -6 & -11 & 3 & 18 \\ 0 & 4 & 7 & 0 \end{array}\right]$$

Exercise 6

Decide which matrices are in row-echelon form:

(a) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 3 & 0 \\ 2 & -5 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 5 & -4 \\ 0 & -2 & 7 \\ 0 & 0 & 0 \end{pmatrix}$

5 Solving Systems Using Augmented Matrices

To solve a system of linear equations using matrices, we do the following steps.

Solving a system using an augmented matrix

To solve a system using matrices:

1. Write the **augmented matrix**.
2. Use row operations to reach **row-echelon form**.
3. Convert back to equations (if you want), then solve by **back-substitution**.

To see an example of Gaussian elimination in matrix form, let's solve the simple system:

$$\begin{cases} 2x - 3y = -2, \\ x + 2y = 13. \end{cases}$$

Our first step is to write the augmented matrix corresponding to this system:

$$\left[\begin{array}{cc|c} 2 & -3 & -2 \\ 1 & 2 & 13 \end{array}\right].$$

Now, we perform three row operations:

$$\left[\begin{array}{cc|c} 2 & -3 & -2 \\ 1 & 2 & 13 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 13 \\ 2 & -3 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 13 \\ 0 & -7 & -28 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{7}R_2} \left[\begin{array}{cc|c} 1 & 2 & 13 \\ 0 & 1 & 4 \end{array}\right].$$

Now that we have obtained a row-echelon form, we can convert the augmented matrix back into the other notation:

$$\left[\begin{array}{cc|c} 1 & 2 & 13 \\ 0 & 1 & 4 \end{array}\right] \longrightarrow \begin{cases} x + 2y = 13, \\ y = 4 \end{cases}$$

Using back substitution, we see immediately that $y = 4$, so:

$$x + 2(4) = 13 \quad \Rightarrow \quad x = 5.$$

The solution is $(5, 4)$.

Exercise 7

Solve the following system (you may use an augmented matrix):

$$\begin{cases} x + y + z = 6, \\ 2x + 3y + z = 11, \\ 3x + 3y + 2z = 16. \end{cases}$$

Solutions to the Exercises

Exercise 1: From $3c = 6$ we get $c = 2$. Substitute into $3b - 2c = 5$:

$$3b - 2(2) = 5 \Rightarrow 3b = 9 \Rightarrow b = 3.$$

Now substitute $b = 3$ and $c = 2$ into $4a + b + c = 9$:

$$4a + 3 + 2 = 9 \Rightarrow 4a = 4 \Rightarrow a = 1.$$

So $(a, b, c) = (1, 3, 2)$.

Exercise 3: (a) 2×3

(b) 2×2

(c) 3×2

Exercise 4: (a) Coefficient matrix:

$$\begin{pmatrix} -1 & 5 \\ 7 & -2 \end{pmatrix} \qquad \text{Augmented matrix: } \left[\begin{array}{cc|c} -1 & 5 & 2 \\ 7 & -2 & -6 \end{array}\right]$$

(b) Coefficient matrix:

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & 0 & 2 \\ -2 & -1 & 0 \end{pmatrix} \qquad \text{Augmented matrix: } \left[\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 1 & 0 & 2 & -3 \\ -2 & -1 & 0 & 4 \end{array}\right]$$

Exercise 5: (a)

$$\left[\begin{array}{ccc|c} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{array}\right]$$

(b)

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{array}\right]$$

(c) $R_3 \to R_3 - 2R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{array}\right]$$

(d) $R_2 \to R_2 + 6R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ 0 & 1 & 15 & -6 \\ 0 & 4 & 7 & 0 \end{array}\right]$$

Exercise 6: (a) Not row-echelon (b) Row-echelon form (c) Row-echelon form.

Exercise 7:

$$(2x + 3y + z) - 2(x + y + z) = 11 - 12 \Rightarrow y - z = -1.$$

$$(3x + 3y + 2z) - 3(x + y + z) = 16 - 18 \Rightarrow -z = -2 \Rightarrow z = 2.$$

Then $y - z = -1$ gives $y - 2 = -1$, so $y = 1$. Finally, $x + y + z = 6$ gives $x + 1 + 2 = 6$, so $x = 3$. So $(x, y, z) = (3, 1, 2)$.