MAT140: Lecture 15 Handout

Determinants

Last lecture we learned how to solve larger systems of linear equations by Gaussian elimination. The basic strategy was simple: take a difficult system, perform row operations until it becomes an easier one, and then solve using back-substitution. We saw that when performing Gaussian elimination, all of the important information was really contained within the coefficients, and therefore we could equally perform elimination in a simpler notation in which we only wrote down these numbers. This gave us *matrices*, which were presented as simple arrays of real numbers.

In this lecture we continue the story of matrices by introducing a special operation called the *determinant*. At first glance, the determinant looks like an algebraic formula attached to a square matrix. However, it turns out to carry a geometric meaning: determinants encode geometric quantities such as area and volume. This is one of our first glimpses of the deeper structure behind linear algebra.

Today we will:

1. Define the determinant of a 2×2 matrix and compute some examples.
2. Learn how to compute determinants of 3×3 matrices using expansion by minors.
3. Use Cramer's rule to test whether a 2×2 system has a unique solution.
4. Interpret determinants geometrically in terms of area, and use them to find the area of a triangle.

1 What is a Determinant?

Up until now, matrices have mostly been a convenient notation that keeps track of the coefficients when solving systems of linear equations. The determinant is our first example of turning a square matrix into a *single number*. This number is surprisingly informative, and has a nice geometric interpretation.

Important restriction

The determinant is only defined for **square matrices**, that is, matrices with the same number of rows and columns.

1.1 The determinant of a 2×2 matrix

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the determinant of A is defined by

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Determinant of a 2×2 matrix

For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det(A) = ad - bc.$$

In words, the determinant is the product of one diagonal minus the product of the other diagonal.

This is a good mnemonic, but it is important to remember the *order*: the determinant is not the sum of the diagonal products, but their *difference*.

1.2 Two quick examples

$$\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = (3)(3) - (2)(1) = 9 - 2 = 7.$$

$$\begin{vmatrix} 7 & 3 \\ 0 & 1 \end{vmatrix} = (7)(1) - (3)(0) = 7 - 0 = 7.$$

Exercise 1

Find the determinant of each matrix.

1. $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$
2. $B = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$
3. $C = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$

1.3 The Determinant of a 3×3 Matrix

The 2×2 determinant is very simple. For a 3×3 matrix, the formula is more complicated, so we break the problem into smaller pieces.

Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

To expand along the first row, we delete the row and column containing each chosen entry, and compute the determinant of the remaining 2×2 matrix. These smaller determinants are called *minors*. The sign pattern is:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

So, along the first row we get

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Expanding a 3×3 determinant along a row

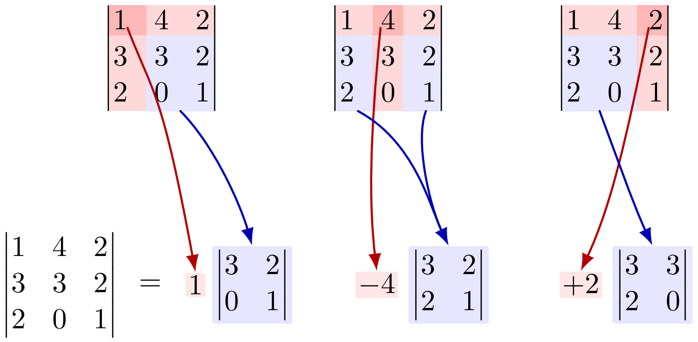
To compute a 3×3 determinant:

1. Pick a row or column.
2. Use the checkerboard sign pattern $+, -, +$ / $-, +, -$ / $+, -, +$.
3. For each chosen entry, delete its row and column to form a 2×2 minor.
4. Multiply each entry by the determinant of its minor, include the correct sign, and add.

As an example, we will compute the determinant:

$$\begin{vmatrix} 1 & 4 & 2 \\ 3 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix}.$$

To do this, we will expand along the first row and create three 2×2 matrices.



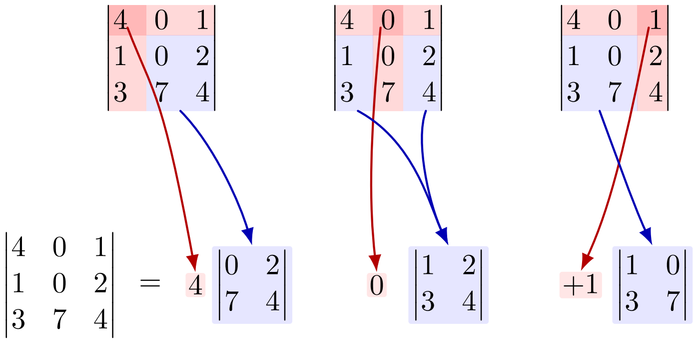
We can now simply compute these minor determinants to get:

$$\begin{aligned} \begin{vmatrix} 1 & 4 & 2 \\ 3 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix} &= 1 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - 4 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 3 \\ 2 & 0 \end{vmatrix} \\ &= 1(3 \cdot 1 - 2 \cdot 0) - 4(3 \cdot 1 - 2 \cdot 2) + 2(3 \cdot 0 - 3 \cdot 2) \\ &= 1(3) - 4(-1) + 2(-6) \\ &= 3 + 4 - 12 = -5. \end{aligned}$$

As another example, we will compute the determinant:

$$\begin{vmatrix} 4 & 0 & 1 \\ 1 & 0 & 2 \\ 3 & 7 & 4 \end{vmatrix}.$$

Schematically, expansion along the first row would look like:



Again, the full determinant can now be computed by calculating the minor determinants:

$$\begin{aligned}\begin{vmatrix}4&0&1\\1&0&2\\3&7&4\end{vmatrix} &= 4\begin{vmatrix}0&2\\7&4\end{vmatrix} - 0\begin{vmatrix}1&2\\3&4\end{vmatrix} + 1\begin{vmatrix}1&0\\3&7\end{vmatrix}\\ &= 4(0\cdot 4 - 2\cdot 7) - 0(1\cdot 4 - 2\cdot 3) + 1(1\cdot 7 - 0\cdot 3)\\ &= 4(-14) - 0 + 7 = -49.\end{aligned}$$

Exercise 2

Compute each determinant.

1. $\begin{vmatrix}1&2&1\\1&2&2\\3&1&2\end{vmatrix}$
2. $\begin{vmatrix}4&0&1\\0&1&2\\0&0&4\end{vmatrix}$
3. $\begin{vmatrix}1&3&1\\2&a&2\\0&0&4\end{vmatrix}$

2 Cramer's Rule

We now return to systems of equations. Earlier in the course we solved systems by graphing, substitution, elimination, and then row reduction. We will now demonstrate that determinants provide a way to solve these systems. We will start with the simplest case of a system of 2 linear equations with 2 unknowns.

2.1 General Solutions to a System of 2 linear equations

Consider the general system:

$$\begin{cases}a_1x + b_1y = c_1,\\ a_2x + b_2y = c_2.\end{cases}$$

where here a_i, b_i and c_i are all real numbers, and for this derivation assume that $b_2 \neq 0$.

We can solve this system generally by using substitution: first rearrange an equation so that the left-hand-side is written purely in terms of y, and then substitute this value into the other equation. For no particular reason, we will rearrange the second equation for y:

$$y = -\frac{a_2}{b_2}x + \frac{c_2}{b_2}.$$

If we now substitute this value into the first equation and rearrange, we obtain:

$$a_1 x + b_1\left(-\frac{a_2}{b_2}x + \frac{c_2}{b_2}\right) = c_1,$$
$$a_1 x - \frac{a_2 b_1}{b_2}x + \frac{b_1 c_2}{b_2} = c_1.$$

Now we can collect the x-terms on the left and constants on the right:

$$a_1 x - \frac{a_2 b_1}{b_2}x + \frac{b_1 c_2}{b_2} = c_1 \quad \Rightarrow \quad a_1 x - \frac{a_2 b_1}{b_2}x = c_1 - \frac{b_1 c_2}{b_2} \quad \Rightarrow \quad \frac{a_1 b_2 - a_2 b_1}{b_2}x = \frac{c_1 b_2 - b_1 c_2}{b_2}.$$

Now we can write the left-hand-side purely in terms of x by dividing:

$$x = \frac{c_1 b_2 - b_1 c_2}{a_1 b_2 - a_2 b_1}.$$

A symmetric argument, this time starting with x instead of y, will yield the expression:

$$y = \frac{c_1 a_2 - a_1 c_2}{b_1 a_2 - b_2 a_1}.$$

Since

$$b_1 a_2 - b_2 a_1 = -(a_1 b_2 - a_2 b_1) \quad \text{and} \quad c_1 a_2 - a_1 c_2 = -(c_2 a_1 - a_2 c_1),$$

this can be rewritten as

$$y = \frac{a_1 c_2 - c_1 a_2}{a_1 b_2 - a_2 b_1}.$$

Therefore, provided that

$$a_1 b_2 - a_2 b_1 \neq 0,$$

the unique solution for this general system will be:

$$x = \frac{c_1 b_2 - b_1 c_2}{a_1 b_2 - a_2 b_1}, \qquad \text{and} \qquad y = \frac{a_1 c_2 - c_1 a_2}{a_1 b_2 - a_2 b_1}.$$

These formulas for the general solutions of x and y look suspiciously like the formula for a determinant of a 2 × 2 matrix. In fact, this is precisely the case. The above two expressions can be rewritten as:

$$x = \frac{\det\begin{pmatrix} c_1 & b_1 \\ c_2 & b_2 \end{pmatrix}}{\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}, \qquad \text{and} \qquad y = \frac{\det\begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}}{\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}.$$

We have just derived Cramer's rule for systems of 2 equations and 2 unknowns.

Cramer's rule for a 2 × 2 system

If

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

then the system

$$\begin{cases} a_1 x + b_1 y = c_1, \\ a_2 x + b_2 y = c_2 \end{cases}$$

has the unique solution

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Importantly, we must assume that the determinant of the coefficient matrix is non-zero. Otherwise, the denominator of our general solution would be zero and therefore undefined. This is quite an important observation.

Uniqueness of Solutions

For a 2 × 2 system, the system has a **unique solution** if and only if the determinant of the coefficient matrix is **nonzero**.

It should be noted that the coefficient matrix having zero determinant does **not** mean that the system has no solution. It merely guarantees that the system does not have a *unique* solution. In that case the system may have no solution, or it may have infinitely-many solutions. For example, consider the following system:

$$\begin{cases} x + y = 1, \\ 2x + 2y = 2 \end{cases}$$

The two equations are describing the same line, so this system has infinitely-many solutions. However, the determinant of the coefficient matrix is:

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = (1 \cdot 2) - (1 \cdot 2) = 0.$$

As an example of Cramer's rule in action, consider the system:

$$\begin{cases} 4x - 2y = 10, \\ 3x - 5y = 11. \end{cases}$$

The coefficient matrix of this system is

$$\begin{pmatrix} 4 & -2 \\ 3 & -5 \end{pmatrix}.$$

This matrix has non-zero determinant:

$$D = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = (4)(-5) - (-2)(3) = -20 - (-6) = -14,$$

so the system has a unique solution. Cramer's rule then gives

$$x = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix}} = \frac{-50+22}{-14} = 2, \qquad \text{and} \quad y = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix}} = \frac{44-30}{-14} = -1.$$

Exercise 3

Use Cramer's rule to determine whether each system has a unique solution.

1. $\begin{cases} x - 3y = 0 \\ x + 3y = 7 \end{cases}$
2. $\begin{cases} 3x - 3y = 0 \\ x - y = 7 \end{cases}$
3. $\begin{cases} 2x - 3y = 0 \\ x + 2 = 7 \end{cases}$

2.2 General Solutions to a System of 3 linear equations

As a matter of fact, there is also a Cramer's rule for systems of 3 linear equations. The derivation is similar to that of before, except that this time we use either Gaussian elimination or an elaborate process of substitution on the general system:

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

For the sake of convenience, we will not perform this derivation. The resulting rule follows.

Cramer's rule for a 3 × 3 system

If

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,$$

then the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

has the unique solution

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Exercise 4

Use Cramer's rule to determine whether or not the following system has a unique solution.

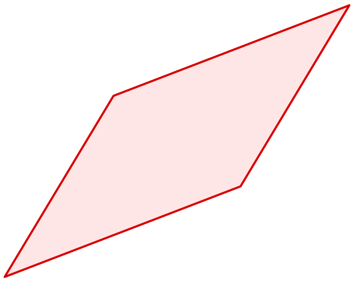
$$\begin{cases} x + y + z = 2, \\ 2x + 2y + 2z = 5, \\ x - y + z = 1 \end{cases}$$

3 The Geometric Meaning of Determinants

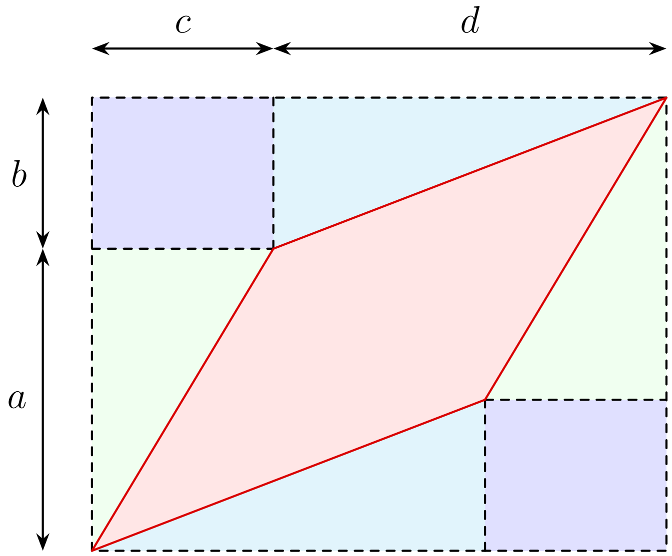
Up to now, determinants have been a purely algebraic device, we computed them from arrays of numbers, and then used them inside formulas for algebraic convenience. We will now see that determinants are *much more* than that, they have a clear and deep geometric meaning.

3.1 Determinants and area

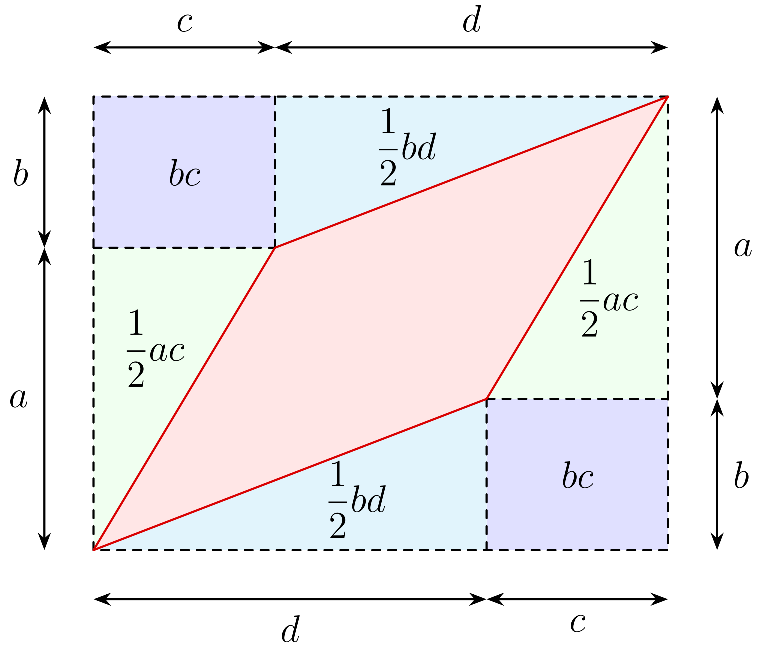
Suppose that we have a parallelogram which is tilted, something like:



How would we find the area of this shape? Ordinarily, the area of a parallelogram is given by its base multiplied by its height. However, in this case the exact description of the length of these two lines may be difficult to write in terms of coordinates x and y, since the parallelogram is tilted slightly. Instead, an easier way to describe the area of the parallelogram would be to create a rectangular box around it, and to project the dimensions of the parallelogram onto the box:



Now, we can describe the area of the parallelogram by taking the total area of the box and removing all of the regions that are not part of the parallelogram, namely the two blue rectangles and the four surrounding triangles. Also, opposite sides of a parallelogram are parallel and equal in length, so the corresponding side lengths on the surrounding box match as shown in the diagram. We can then use the area formulae for rectangles and triangles to describe the various areas inside our box:



We now observe that the area of the total box equals $(a+b)(c+d)$. Therefore the area of the parallelogram is:

$$\begin{aligned} Area &= (a+b)(c+d) - 2\left(bc + \frac{1}{2}bd + \frac{1}{2}ac\right) \\ &= (a+b)(c+d) - 2bc - bd - ac \\ &= ac + ad + bc + bd - 2bc - bd - ac \\ &= ad - bc. \end{aligned}$$

Now, something remarkable has just happened: we have just obtained the formula for the determinant of a 2×2 matrix. Specifically, we have just demonstrated that the area of a parallelogram is given by

$$\pm \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm(ad - bc),$$

where a, b, c and d are the lengths described in the diagram above. In other words, generally speaking, the determinant of a 2×2 matrix gives the area of the corresponding parallelogram up to sign.

Determinants as area

For a 2×2 matrix A, the area of the corresponding parallelogram is given by $\pm\det(A)$, where here the ± sign means that we take the positive value.
So:

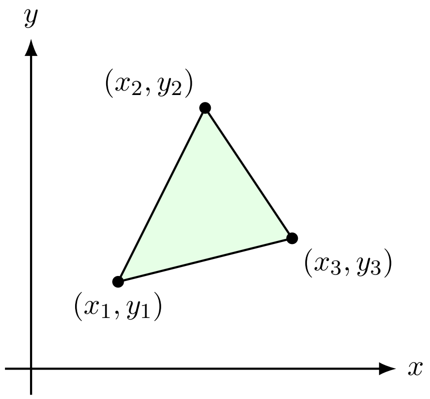
- $\det(A) \neq 0$ means the shape has nonzero area,
- $\det(A) = 0$ means the shape has collapsed and its area is zero.

3.2 The area of a triangle

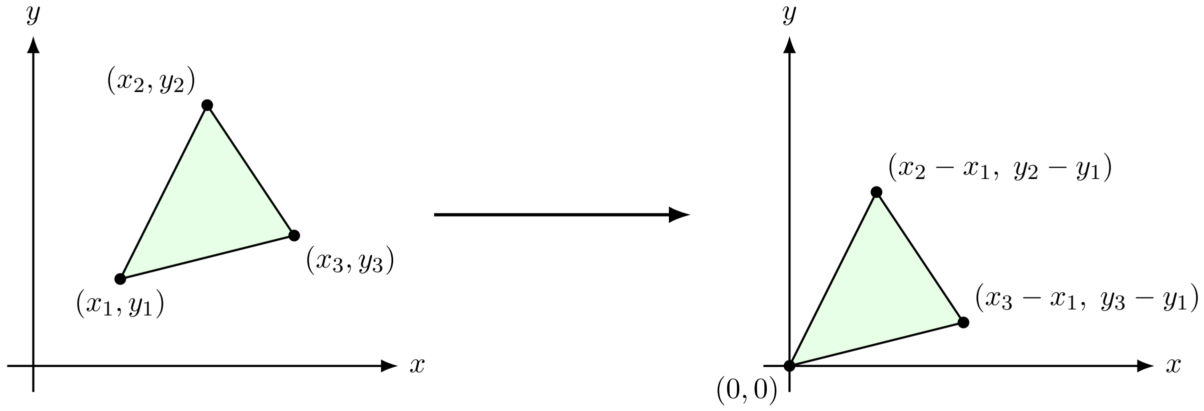
Suppose now that we have a triangle in the $2D$ plane, with vertices

$$(x_1, y_1), \qquad (x_2, y_2), \qquad (x_3, y_3).$$

Graphically, this could be:



We can calculate the area of this triangle by first translating it so that one of the vertices coincides with the origin:



After this translation, the two remaining vertices determine the two side vectors of a parallelogram. So, in the notation of the previous section, we may take

$$a = x_2 - x_1, \qquad b = x_3 - x_1, \qquad c = y_2 - y_1, \qquad d = y_3 - y_1.$$

The area of the corresponding parallelogram is therefore

$$\pm \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} = \pm\big((x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\big).$$

Expanding this expression gives:

$$\pm\big(x_1(y_2 - y_3) - y_1(x_2 - x_3) + x_2y_3 - x_3y_2\big).$$

The area of the triangle will then be half of this quantity. Another way to write this quantity is with a 3 × 3 determinant:

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

We can double check this by expanding this determinant along the first row. If we do that, we recover our previous formula:

$$\text{Area} = \pm\frac{1}{2}\big(x_1(y_2 - y_3) - y_1(x_2 - x_3) + x_2y_3 - x_3y_2\big).$$

Area of a triangle

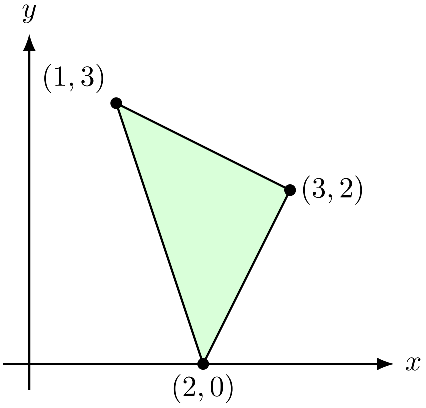
If the vertices of a triangle are (x_1, y_1), (x_2, y_2), and (x_3, y_3), then the area of this triangle is given by

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

where here the ± sign is shorthand for choosing the positive value of the resulting expression.

Exercise 5

Find the area of the triangle with vertices $(2,0)$, $(1,3)$, and $(3,2)$.



3.3 A brief remark on volume

The same story continues in three dimensions. Determinants are not only about area in the plane, but also about volume in space. In particular, if a tetrahedron has vertices

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3), \quad (x_4, y_4, z_4),$$

then its volume is given by

$$\text{Volume} = \pm\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

We will not work deeply with this formula yet. For now, the important point is the analogy:

$$\text{determinant in 2D} \longleftrightarrow \text{area}, \qquad \text{determinant in 3D} \longleftrightarrow \text{volume}.$$

Solutions to the Exercises

Exercise 1

$$\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 2(4) - (-3)(1) = 8 + 3 = 11.$$

$$\det(B) = \begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} = (-1)(-4) - (2)(2) = 4 - 4 = 0.$$

$$\det(C) = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = (1)(5) - (3)(2) = 5 - 6 = -1.$$

Exercise 2

1.

$$\begin{vmatrix}1 & 2 & 1\\ 1 & 2 & 2\\ 3 & 1 & 2\end{vmatrix} = 1\begin{vmatrix}2 & 2\\ 1 & 2\end{vmatrix} - 2\begin{vmatrix}1 & 2\\ 3 & 2\end{vmatrix} + 1\begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix}$$

$$= 1(4-2) - 2(2-6) + 1(1-6)$$

$$= 2 + 8 - 5 = 5.$$

2.

$$\begin{vmatrix}4 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{vmatrix} = 4\begin{vmatrix}1 & 2\\ 0 & 4\end{vmatrix} - 0\begin{vmatrix}0 & 2\\ 0 & 4\end{vmatrix} + 1\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix}$$

$$= 4(4) - 0 + 1(0)$$

$$= 16.$$

3.

$$\begin{vmatrix}1 & 3 & 1\\ 2 & a & 2\\ 0 & 0 & 4\end{vmatrix} = 1\begin{vmatrix}a & 2\\ 0 & 4\end{vmatrix} - 3\begin{vmatrix}2 & 2\\ 0 & 4\end{vmatrix} + 1\begin{vmatrix}2 & a\\ 0 & 0\end{vmatrix}$$

$$= 1(4a) - 3(8) + 1(0)$$

$$= 4a - 24 = 4(a-6).$$

Exercise 3

1. The coefficient matrix is

$$\begin{pmatrix}1 & -3\\ 1 & 3\end{pmatrix}, \qquad \det = \begin{vmatrix}1 & -3\\ 1 & 3\end{vmatrix} = 3 - (-3) = 6.$$

Since the determinant is nonzero, the system has a unique solution.

2. The coefficient matrix is

$$\begin{pmatrix}3 & -3\\ 1 & -1\end{pmatrix}, \qquad \det = \begin{vmatrix}3 & -3\\ 1 & -1\end{vmatrix} = -3 - (-3) = 0.$$

So the system does not have a unique solution.

3. Rewrite the second equation as $x + 0y = 5$. Then the coefficient matrix is

$$\begin{pmatrix}2 & -3\\ 1 & 0\end{pmatrix}, \qquad \det = \begin{vmatrix}2 & -3\\ 1 & 0\end{vmatrix} = 0 - (-3) = 3.$$

Since the determinant is nonzero, the system has a unique solution.

Exercise 4

The coefficient matrix is

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 1 & -1 & 1\end{pmatrix}, \qquad D = \begin{vmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 1 & -1 & 1\end{vmatrix}.$$

Expanding along the first row gives

$$\begin{aligned} D &= 1\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} \\ &= 1(2 \cdot 1 - 2(-1)) - 1(2 \cdot 1 - 2 \cdot 1) + 1(2(-1) - 2 \cdot 1) \\ &= 4 - 0 - 4 = 0. \end{aligned}$$

Since the determinant of the coefficient matrix is zero, the system does not have a unique solution.

Exercise 5

Use

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix}.$$

Expand along the first row:

$$\begin{aligned} \begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix} &= 2\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} \\ &= 2(3 \cdot 1 - 1 \cdot 2) + 1(1 \cdot 2 - 3 \cdot 3) \\ &= 2(1) + (-7) = -5. \end{aligned}$$

Therefore

$$\text{Area} = \pm\frac{1}{2}(-5) = \frac{5}{2}.$$