

MAT140: Lecture 17 Handout

Radicals

Last lecture we completed our discussion of matrices, determinants, and linear algebra. We will now move on to the final topic of our course, by discussing polynomials and their roots in depth.

In this lecture we start our discussion by introducing *radicals*, which are the inverse operation to exponentiation. At first glance radicals can look familiar, and conceptually they are not so strange. However, these first familiar steps may end up leading to a very unusual direction over time.

Today we will:

1. Define n th roots and principal roots, and explain when radicals are real.
2. Connect radicals with rational exponents and the inverse properties of powers and roots.
3. Simplify radicals by removing perfect-power factors, including expressions with variables.
4. Add, subtract, multiply, and divide radicals, including the use of conjugates.
5. Apply radicals to right triangles, and briefly preview radical equations.

1 Roots and Radicals

1.1 The basic idea

A radical is another notation for a root, such as a square root. If $b^n = a$, then b is called an n th root of a . When the principal n th root exists, we write it as $\sqrt[n]{a}$. Thus:

$$b^n = a \quad \implies \quad b \text{ is an } n\text{th root of } a.$$

When $n = 2$, we usually write \sqrt{a} instead of $\sqrt[2]{a}$, and we call this the *square root* of a . When $n = 3$, we call $\sqrt[3]{a}$ the *cube root* of a . For any $n > 3$, we simply refer to $\sqrt[n]{a}$ as the n th root.

Definition of an n th root

Let $n \geq 2$ be an integer. If $b^n = a$, then b is called an n th root of a . Here n is called the *index* of the radical, and a is called the *radicand*.

For example, $2^2 = 4$, so 2 is a square root of 4, and $3^4 = 81$, so 3 is a fourth root of 81.

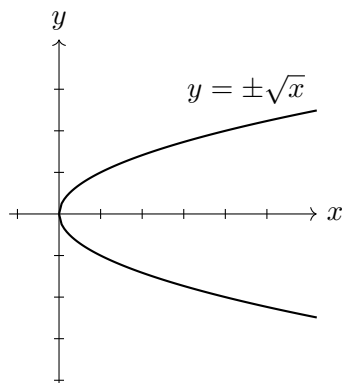
1.2 Principal roots and the even/odd distinction

We now need to make one subtle yet important convention. The symbol \sqrt{a} means the *principal square root*, that is, the nonnegative square root of a . For instance, both 6 and -6 square to 36, but by convention

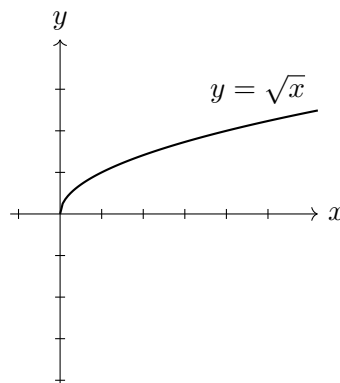
$$\sqrt{36} = 6,$$

not -6 . Ultimately, we do this because we want to be able to treat $\sqrt{\cdot}$ as a function, that way, it can be included in algebraic manipulations with ease. However, if we were to include the negative root and write something like $\sqrt{36} = \pm 6$, then this would no longer be a function, since one input

(36 in this case) would have two distinct outputs (+6 and -6). Selecting the principal root allows us to effectively see a square root as a function:



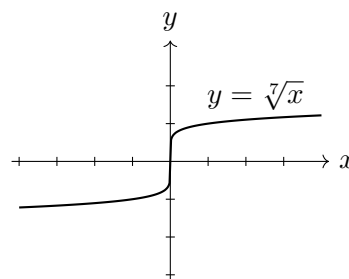
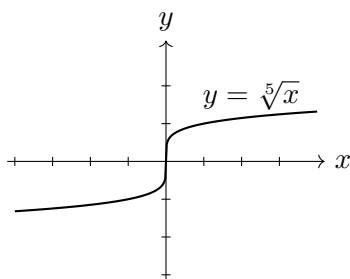
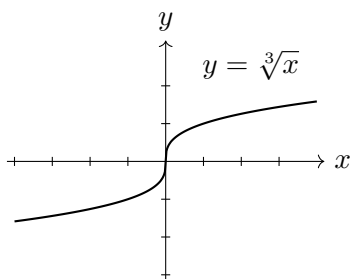
The “naive” square root



Selecting the Principal Root

As you can see, the naive square root fails the vertical line test. So, to repair this, we “chop off” the negative part of the square root, and this allows us to interpret \sqrt{x} as a genuine function. Observe in the graphs above, there is no value for the square root of a negative number. This is because the product of two negative numbers is positive: for example, $-3 \cdot -3 = 9$, but no real number multiplied by itself will ever equal -9 . Therefore, when viewed as a real-valued function, \sqrt{x} is defined only for $x \geq 0$. In cases like $\sqrt{-9}$ there is simply *no real number* that corresponds to it.

Interestingly, our convention of selecting principal roots is only necessary for radicals containing *even* roots. For radicals containing *odd* roots, the radicals $\sqrt[n]{x}$ are already genuine functions:



Fundamentally, this is because two negative numbers always multiply to make a positive number. Recall that exponentiation is just a compact notation for repeated multiplication:

$$b^n = a \quad \Leftrightarrow \quad b \cdot b \cdot \dots \cdot b = a.$$

So, if we write $b = \sqrt[n]{a}$, we are really saying “ b multiplied by itself n times in a row will equal a ”. If n is *even*, then $b^n = (-b)^n$ because the minus signs will cancel each other out in pairs. If n is *odd* then these minus signs will not cancel each other out, and there will be one left over. This means:

$$\text{if } a = b^n \text{ then:} \quad (-b)^n = (-b) \cdot (-b) \cdot \dots \cdot (-b) = \begin{cases} a & \text{if } n \text{ is even} \\ -a & \text{if } n \text{ is odd} \end{cases}.$$

This is one of the first places where even and odd roots behave differently: we can take n th roots of negative numbers only when n is odd.

Even and odd roots

1. If n is odd, then every real number a has exactly one real n th root.
2. If n is even and $a > 0$, then a has two real n th roots, namely $\pm \sqrt[n]{a}$.
3. If n is even and $a = 0$, then a has exactly one real n th root, namely 0.
4. If n is even and $a < 0$, then a has no real n th root.
5. The symbol $\sqrt[n]{a}$ denotes the principal root.

Exercise 1

Find each root, if it is real.

1. $\sqrt{36}$
2. $-\sqrt{36}$
3. $\sqrt{-4}$
4. $\sqrt[3]{8}$
5. $\sqrt[3]{-8}$

2 Inverse Properties and Rational Exponents

2.1 Powers and roots are inverses of each other

Radicals are the inverse operation to exponents, but we must be careful about signs. If the index n is odd, then

$$\sqrt[n]{a^n} = a.$$

If the index n is even, then the principal root must be nonnegative, so

$$\sqrt[n]{a^n} = |a|.$$

For example, take $a = -2$ and $n = 2$:

$$\sqrt{(-2)^2} = \sqrt{4} = 2 = |-2|.$$

On the other hand, if a already has a principal n th root, then raising that root back to the n th power returns a :

$$(\sqrt[n]{a})^n = a.$$

Inverse properties

For real a and integer $n \geq 2$:

$$(\sqrt[n]{a})^n = a$$

whenever the principal n th root exists. Also,

$$\sqrt[n]{a^n} = \begin{cases} a, & \text{if } n \text{ is odd,} \\ |a|, & \text{if } n \text{ is even.} \end{cases}$$

For example,

$$(\sqrt{4})^2 = 4, \quad \sqrt[3]{27} = 3, \quad \sqrt[4]{16} = 2, \quad \sqrt[5]{-243} = -3.$$

Exercise 2

Evaluate each expression.

1. $(\sqrt{81})^2$
2. $\sqrt[3]{64}$
3. $\sqrt[5]{32}$
4. $\sqrt[3]{-8}$

2.2 Rational exponents

Radicals can also be written using fractional exponents. The rule is:

$$a^{1/n} = \sqrt[n]{a}.$$

This naturally comes about due to the inverse relationship with powers. Observe that:

$$(a^{\frac{1}{n}})^n = a^{\frac{1}{n} \cdot n} = a^1 = a$$

so raising a number to the power of $\frac{1}{n}$ is the inverse operation of raising it to the power of n . Since inverse operations have to be unique, it must be the case that $\sqrt[n]{a}$ and $a^{\frac{1}{n}}$ are the same.

More generally, we have that:

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

This perspective is extremely helpful, because it allows us to use the usual laws of exponents when simplifying radical expressions.

Rational exponents

If the relevant roots are real, then

$$a^{1/n} = \sqrt[n]{a}, \quad a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

For example,

$$8^{4/3} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16,$$

and

$$25^{-3/2} = \frac{1}{25^{3/2}} = \frac{1}{(\sqrt{25})^3} = \frac{1}{5^3} = \frac{1}{125}.$$

Note that the order of operations still matters. In particular, exponentiation occurs *before* multiplication, so minus signs interact with exponents as:

$$-16^{1/2} = -(16^{1/2}) = -4,$$

but parentheses come first, so:

$$(-16)^{1/2} = \sqrt{-16},$$

which is not a real number.

Exercise 3

Evaluate each expression, working over the real numbers.

1. $8^{4/3}$
2. $(4^2)^{3/2}$
3. $25^{-3/2}$
4. $\left(\frac{64}{125}\right)^{2/3}$
5. $-16^{1/2}$
6. $(-16)^{1/2}$

3 Simplifying Radicals

3.1 Removing factors

A radical expression is simpler when we have removed as many factors as possible from the radicand. In order to do this, we can use one of the following equalities.

Product and quotient rules for radicals

Whenever the radicals involved are real,

$$\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}, \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad (b \neq 0).$$

As a matter of fact, these equalities follow directly from the rules of exponents. For the first equality, we may rewrite the n th root as the rational exponent $\frac{1}{n}$, giving:

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = \left(a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}\right)^{n \cdot \frac{1}{n}} = \left(a^{\frac{1}{n} \cdot n} \cdot b^{\frac{1}{n} \cdot n}\right)^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{ab}.$$

The quotient rule mentioned above also follows from the rules of exponents. We have:

$$\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

A radical of the form $\sqrt[n]{ab}$ can then be simplified if a (or b) is of the form c^n . For example, if it were the case that $a = c^n$, then we could simplify the radical by writing:

$$\sqrt[n]{ab} = \sqrt[n]{c^n \cdot b} = \sqrt[n]{c^n} \cdot \sqrt[n]{b} = \begin{cases} c \sqrt[n]{b}, & \text{if } n \text{ is odd,} \\ |c| \sqrt[n]{b}, & \text{if } n \text{ is even,} \end{cases}$$

where here we have made use of the inverse property $\sqrt[n]{c^n} = c$ when n is odd, and $\sqrt[n]{c^n} = |c|$ when n is even.

Our strategy for simplifying radicals is therefore to ask “does this number break down into a product with an n th power of something?” For example, consider $\sqrt{75}$. Here, we notice that $75 = 25 \times 3$,

and it just so happens that $25 = 5^2$. Therefore, we can simplify the radical $\sqrt{75}$ by rewriting it using the product rule:

$$\begin{aligned}\sqrt{75} &= \sqrt{25 \cdot 3} = \sqrt{5^2 \cdot 3} = \sqrt{5^2} \cdot \sqrt{3} = 5\sqrt{3}. \\ \sqrt{72} &= \sqrt{36 \cdot 2} = 6\sqrt{2},\end{aligned}$$

and

$$\sqrt{162} = \sqrt{81 \cdot 2} = 9\sqrt{2}.$$

It is helpful to remember the following numbers:

- Square numbers: 1, 4, 9, 16, 25, 36, 49, ...
- Cube numbers: 1, 8, 27, 64, 125, ...
- Fourth powers: 1, 16, 81, 256, ...

3.2 Variables inside radicals

When variables appear inside square roots, the absolute value issue becomes important. For example,

$$\sqrt{x^2} = |x|,$$

rather than simply x . This is because x is a *variable*, and it is allowed to take on values that are negative. So, when we simplify radicals involving variables, we have to be careful and keep track of the sign of the variable. For example:

$$\sqrt{25x^2} = 5\sqrt{x^2} = 5|x|.$$

It is worth mentioning that sometimes the expression itself already forces a variable to be positive. In cases like this, writing the absolute value sign would be redundant. To see this demonstrated, consider the radical expression $\sqrt{12x^3}$. We may simplify this as per usual:

$$\sqrt{12x^3} = \sqrt{4x^2 \cdot 3x} = 2\sqrt{x^2}\sqrt{3x} = 2|x|\sqrt{3x}.$$

If we are working over the real numbers, the expression $\sqrt{12x^3}$ *already forces* $x \geq 0$, because we cannot take a square root of a negative number. Since the absolute value of a positive number is itself, we may simplify further to

$$\sqrt{12x^3} = 2x\sqrt{3x}.$$

Similarly,

$$\sqrt{144x^4} = 12\sqrt{x^4} = 12x^2,$$

since $x^2 \geq 0$.

Simplest radical form

A radical expression is usually considered to be in simplest form when:

1. no perfect n th-power factor remains inside the radical,
2. no denominator contains a radical,
3. any like radicals have been combined.

Exercise 4

Simplify each expression as much as possible.

1. $\sqrt{75}$
2. $\sqrt{72}$
3. $\sqrt{162}$
4. $\sqrt{25x^2}$
5. $\sqrt{12x^3}$
6. $\sqrt{144x^4}$
7. $\sqrt{72x^3y^2}$

3.3 Rationalising denominators

A radical is not considered fully simplified if it remains in the denominator of a fraction. The process of removing radicals from denominators is called *rationalising the denominator*. The strategy here is to use three familiar properties of real numbers:

1. Multiplicative Identity: $a \cdot 1 = a$ for all real numbers a
2. Multiplicative Inverse: $a \cdot \frac{1}{a} = \frac{a}{a} = 1$ for all non-zero real numbers a .
3. Equivalent Fractions: multiplying the numerator and denominator of a fraction by the same non-zero quantity does not change its value.

To remove a radical from the denominator of a fraction, we may follow the process:

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot 1 = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{1 \cdot \sqrt{a}}{(\sqrt{a})^2} = \frac{\sqrt{a}}{a},$$

for all $a > 0$. However, it should be noted that this only works because there is a lone radical in the denominator, we will deal with the more-complicated cases later. As an example of rationalising a denominator, we simplify:

$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{3 \cdot 5}}{\sqrt{5 \cdot 5}} = \frac{\sqrt{15}}{5}.$$

This process also works for other radicals with higher roots. For example:

$$\frac{4}{\sqrt[3]{9}} = \frac{4}{\sqrt[3]{9}} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{3}} = \frac{4\sqrt[3]{3}}{\sqrt[3]{27}} = \frac{4\sqrt[3]{3}}{3}.$$

And,

$$\frac{8}{9\sqrt{2}} = \frac{8}{9\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{8\sqrt{2}}{18} = \frac{4\sqrt{2}}{9}.$$

Exercise 5

Rationalise each denominator and simplify.

1. $\sqrt{\frac{2}{7}}$
2. $\frac{2}{\sqrt[3]{4}}$
3. $\frac{5}{2\sqrt{3}}$

4 Adding, Subtracting, Multiplying, and Dividing Radicals

4.1 Like radicals

Two radicals are called *like radicals* if they have the same index and the same radicand. These can be combined by adding or subtracting their coefficients, just as like terms are combined in algebra.

For example,

$$\sqrt{7} + 5\sqrt{7} - 2\sqrt{7} = (1 + 5 - 2)\sqrt{7} = 4\sqrt{7}.$$

But

$$\sqrt{2} + \sqrt{3}$$

cannot be simplified by addition, because the radicals are not alike.

Addition of radicals

$$a\sqrt[n]{m} + b\sqrt[n]{m} = (a + b)\sqrt[n]{m}.$$

This only works when both radicals have the same index and the same radicand.

For example,

$$6\sqrt{x} - \sqrt[3]{4} - 5\sqrt{x} + 2\sqrt[3]{4} = (6 - 5)\sqrt{x} + (-1 + 2)\sqrt[3]{4} = \sqrt{x} + \sqrt[3]{4},$$

and

$$3\sqrt[3]{x} + 2\sqrt[3]{x} + \sqrt{x} - 8\sqrt{x} = (3 + 2)\sqrt[3]{x} + (1 - 8)\sqrt{x} = 5\sqrt[3]{x} - 7\sqrt{x}.$$

Exercise 6

Simplify each expression.

1. $\sqrt{7} + 5\sqrt{7} - 2\sqrt{7}$
2. $6\sqrt{x} - \sqrt[3]{4} - 5\sqrt{x} + 2\sqrt[3]{4}$
3. $3\sqrt[3]{x} + 2\sqrt[3]{x} + \sqrt{x} - 8\sqrt{x}$
4. A triangle has side lengths \sqrt{x} , $3\sqrt{x}$, and $\sqrt{10x}$. Write and simplify an expression for its perimeter.

4.2 Multiplication and conjugates

Multiplication is often easier than addition, because we can multiply the radicands directly using the product rule of Section 3. For example:

$$\sqrt{6} \cdot \sqrt{3} = \sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2},$$

and similarly:

$$\sqrt[3]{5} \cdot \sqrt[3]{16} = \sqrt[3]{80} = \sqrt[3]{8 \cdot 10} = \sqrt[3]{8} \cdot \sqrt[3]{10} = 2\sqrt[3]{10}.$$

A particularly important idea is that of *conjugates* of radicals.

Conjugates

If two expressions differ only by the sign in the middle, then they are conjugates:

$$a + \sqrt{b} \quad \text{and} \quad a - \sqrt{b}.$$

Their product is

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 + a\sqrt{b} - a\sqrt{b} - (\sqrt{b})^2 = a^2 - b.$$

Observe that when we multiply an expression $a + \sqrt{b}$ by its conjugate, the result is an expression that does not contain any radical. This makes conjugates extremely useful when rationalising denominators. For example:

$$\frac{3}{1 - \sqrt{5}} = \frac{3}{1 - \sqrt{5}} \cdot \frac{1 + \sqrt{5}}{1 + \sqrt{5}} = \frac{3(1 + \sqrt{5})}{1 - 5} = -\frac{3 + 3\sqrt{5}}{4},$$

and

$$\frac{4}{2 - \sqrt{3}} = \frac{4}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{4(2 + \sqrt{3})}{4 - 3} = 8 + 4\sqrt{3}.$$

Exercise 7

Compute each expression and simplify.

1. $\sqrt{6} \cdot \sqrt{3}$
2. $\sqrt[3]{5} \cdot \sqrt[3]{16}$
3. Find the conjugate of $2 - \sqrt{5}$, then multiply the two conjugates together.
4. Rationalise the denominator of $\frac{3}{1 - \sqrt{5}}$.
5. Rationalise the denominator of $\frac{4}{2 - \sqrt{3}}$.

5 The Pythagorean Theorem

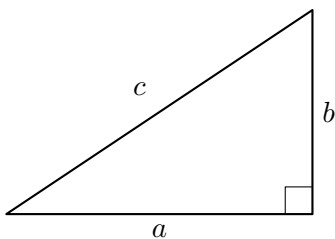
5.1 Right triangles and the Pythagorean theorem

Radicals appear naturally in geometry. If a right triangle has side lengths a and b with hypotenuse c , then the Pythagorean theorem says that:

$$c^2 = a^2 + b^2,$$

so:

$$c = \sqrt{a^2 + b^2}.$$



For example, if $a = 6$ and $b = 9$, then

$$c = \sqrt{6^2 + 9^2} = \sqrt{36 + 81} = \sqrt{117} = \sqrt{9 \cdot 13} = 3\sqrt{13}.$$

Exercise 8

1. Find the hypotenuse of a right triangle whose legs have lengths 6 and 9.
2. Solve the radical equation

$$\sqrt{x + 5} = 4.$$

Check your answer.

Solutions to the Exercises

Exercise 1

1. $\sqrt{36} = 6$.
2. $-\sqrt{36} = -6$.
3. $\sqrt{-4}$ is not a real number.
4. $\sqrt[3]{8} = 2$.
5. $\sqrt[3]{-8} = -2$.

Exercise 2

1. $(\sqrt{81})^2 = 9^2 = 81$.
2. $\sqrt[3]{64} = 4$.
3. $\sqrt[5]{32} = 2$.
4. $\sqrt[3]{-8} = -2$.

Exercise 3

1.

$$8^{4/3} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16.$$

2.

$$(4^2)^{3/2} = 4^{2 \cdot 3/2} = 4^3 = 64.$$

3.

$$25^{-3/2} = \frac{1}{25^{3/2}} = \frac{1}{(\sqrt{25})^3} = \frac{1}{125}.$$

4.

$$\left(\frac{64}{125}\right)^{2/3} = \frac{64^{2/3}}{125^{2/3}} = \frac{(\sqrt[3]{64})^2}{(\sqrt[3]{125})^2} = \frac{4^2}{5^2} = \frac{16}{25}.$$

5.

$$-16^{1/2} = -(16^{1/2}) = -4.$$

6.

$$(-16)^{1/2} = \sqrt{-16},$$

which is not a real number.

Exercise 4

1.

$$\sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}.$$

2.

$$\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}.$$

3.

$$\sqrt{162} = \sqrt{81 \cdot 2} = 9\sqrt{2}.$$

4.

$$\sqrt{25x^2} = 5\sqrt{x^2} = 5|x|.$$

5.

$$\sqrt{12x^3} = \sqrt{4x^2 \cdot 3x} = 2\sqrt{x^2}\sqrt{3x} = 2|x|\sqrt{3x}.$$

Over the reals, this may also be written as $2x\sqrt{3x}$, since $x \geq 0$.

6.

$$\sqrt{144x^4} = 12\sqrt{x^4} = 12x^2.$$

7.

$$\sqrt{72x^3y^2} = \sqrt{36x^2y^2 \cdot 2x} = 6\sqrt{x^2}\sqrt{y^2}\sqrt{2x} = 6|x||y|\sqrt{2x}.$$

In the common case $x \geq 0$, this may also be written as $6x|y|\sqrt{2x}$.

Exercise 5

1.

$$\sqrt{\frac{2}{7}} = \frac{\sqrt{2}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{14}}{7}.$$

2.

$$\frac{2}{\sqrt[3]{4}} = \frac{2}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{2\sqrt[3]{2}}{\sqrt[3]{8}} = \frac{2\sqrt[3]{2}}{2} = \sqrt[3]{2}.$$

3.

$$\frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{2 \cdot 3} = \frac{5\sqrt{3}}{6}.$$

Exercise 6

1.

$$\sqrt{7} + 5\sqrt{7} - 2\sqrt{7} = (1 + 5 - 2)\sqrt{7} = 4\sqrt{7}.$$

2.

$$6\sqrt{x} - \sqrt[3]{4} - 5\sqrt{x} + 2\sqrt[3]{4} = (6 - 5)\sqrt{x} + (-1 + 2)\sqrt[3]{4} = \sqrt{x} + \sqrt[3]{4}.$$

3.

$$3\sqrt[3]{x} + 2\sqrt[3]{x} + \sqrt{x} - 8\sqrt{x} = (3 + 2)\sqrt[3]{x} + (1 - 8)\sqrt{x} = 5\sqrt[3]{x} - 7\sqrt{x}.$$

4.

$$P = \sqrt{x} + 3\sqrt{x} + \sqrt{10x} = (1 + 3)\sqrt{x} + \sqrt{10}\sqrt{x} = (4 + \sqrt{10})\sqrt{x}.$$

Exercise 7

1.

$$\sqrt{6} \cdot \sqrt{3} = \sqrt{18} = 3\sqrt{2}.$$

2.

$$\sqrt[3]{5} \cdot \sqrt[3]{16} = \sqrt[3]{80} = \sqrt[3]{8 \cdot 10} = 2\sqrt[3]{10}.$$

3. The conjugate of $2 - \sqrt{5}$ is $2 + \sqrt{5}$, and

$$(2 - \sqrt{5})(2 + \sqrt{5}) = 2^2 - (\sqrt{5})^2 = 4 - 5 = -1.$$

4.

$$\frac{3}{1 - \sqrt{5}} = \frac{3}{1 - \sqrt{5}} \cdot \frac{1 + \sqrt{5}}{1 + \sqrt{5}} = \frac{3(1 + \sqrt{5})}{1 - 5} = -\frac{3 + 3\sqrt{5}}{4}.$$

5.

$$\frac{4}{2 - \sqrt{3}} = \frac{4}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{4(2 + \sqrt{3})}{4 - 3} = 8 + 4\sqrt{3}.$$

Exercise 8

1.

$$c = \sqrt{6^2 + 9^2} = \sqrt{36 + 81} = \sqrt{117} = \sqrt{9 \cdot 13} = 3\sqrt{13}.$$

2.

$$\sqrt{x + 5} = 4.$$

Square both sides:

$$x + 5 = 16,$$

so

$$x = 11.$$

Check:

$$\sqrt{11 + 5} = \sqrt{16} = 4,$$

so $x = 11$ is correct.