MAT140: Lecture 18 Handout

Complex Numbers

Last lecture we studied radicals, rational exponents, conjugates, and equations involving square roots. In particular, we saw that radicals are the inverse operation to powers, and that conjugates are useful for simplifying certain expressions. This lecture begins from those ideas and asks what happens when they are pushed slightly further than the real numbers allow.

A first problem appears immediately. Over the real numbers, expressions such as $\sqrt{-4}$ do not exist. However, certain algebraic problems naturally lead us towards such quantities anyway. Historically, this happened in the sixteenth century in the work of Gerolamo Cardano, whose attempts to solve cubic equations forced mathematicians to confront square roots of negative numbers. What first looked impossible eventually led to one of the most useful extensions of the number system.

In this lecture we will introduce imaginary numbers and complex numbers, explain how to perform arithmetic with them, and then interpret them geometrically in the plane. In this sense, the lecture sits somewhere between algebra and geometry. On the one hand, complex numbers are new algebraic objects with their own arithmetic. On the other hand, they can also be pictured as points or vectors in a plane, so many familiar geometric ideas reappear in a new form.

Today we will:

1. Briefly review radicals, conjugates, and radical equations.
2. Explain how Cardano's cubic formula motivates square roots of negative numbers.
3. Define the imaginary unit i, and rewrite negative square roots in i-form.
4. Define complex numbers $a + bi$, and describe equality, addition, subtraction, multiplication, and division.
5. Introduce complex conjugates and the Argand plane, and interpret $|a + bi|$ geometrically.

1 A Short Recap of Radicals

Last lecture we used three basic ideas repeatedly:

1. radicals are inverse operations to powers,
2. radicals can be written using rational exponents,
3. conjugates help simplify radical fractions.

For instance,

$$(\sqrt{x})^2 = x, \qquad \sqrt{x^2} = |x|, \qquad \sqrt{x} = x^{1/2},$$

and

$$\sqrt{x}\sqrt{y} = \sqrt{xy}$$

whenever both sides are real.

We also saw that a denominator containing a square root can often be simplified by multiplying by the conjugate. For example,

$$\frac{\sqrt{3}}{1-\sqrt{5}} = \frac{\sqrt{3}}{1-\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} = -\frac{\sqrt{3}+\sqrt{15}}{4},$$

and

$$\frac{4}{2-\sqrt{3}} = \frac{4}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} = 8 + 4\sqrt{3}.$$

Finally, because radicals undo powers, radical equations can often be solved by isolating the radical and then raising both sides to the relevant power. For example,

$$\sqrt{2x-8} = 4$$

gives

$$2x - 8 = 16 \qquad \Longrightarrow \qquad 2x = 24 \qquad \Longrightarrow \qquad x = 12.$$

Reminder

The symbol $\sqrt{a}$ denotes the principal square root of a. Over the real numbers, $\sqrt{a}$ only exists when $a \geq 0$.

Exercise 1

Evaluate or simplify each expression.

1. $\dfrac{\sqrt{2}}{1-\sqrt{3}}$
2. $\dfrac{5}{3-\sqrt{5}}$
3. Solve $\sqrt{3x+1} = 5$

2 Cardano and the Appearance of the Impossible

A major historical reason for introducing complex numbers came from attempts to solve cubic equations. Cardano studied equations of the form

$$x^3 = 3px + 2q,$$

where p and q are real numbers. He found that one solution could be written as

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

We will not derive this formula here. Instead, what matters for us are the consequences.

2.1 A friendly example

If $p = 1$ and $q = 1$, then the equation becomes

$$x^3 = 3x + 2.$$

Cardano's formula gives

$$x = \sqrt[3]{1 + \sqrt{1^2 - 1^3}} + \sqrt[3]{1 - \sqrt{1^2 - 1^3}} = \sqrt[3]{1} + \sqrt[3]{1} = 2.$$

Indeed,

$$2^3 = 8 \qquad \text{and} \qquad 3(2) + 2 = 8.$$

2.2 A stranger example

Now take $p = 5$ and $q = 2$. Then the equation becomes

$$x^3 = 15x + 4.$$

Cardano's formula gives

$$x = \sqrt[3]{2 + \sqrt{2^2 - 5^3}} + \sqrt[3]{2 - \sqrt{2^2 - 5^3}} = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

At first glance this seems impossible over the real numbers, because $\sqrt{-121}$ is not a real number. Yet the original equation certainly has real solutions. In fact, one can check that

$$x = 4, \qquad x = -2 + \sqrt{3}, \qquad x = -2 - \sqrt{3}$$

all satisfy

$$x^3 = 15x + 4.$$

So something remarkable has happened: a formula for a *real* equation has led us through square roots of negative numbers. This is the point at which mathematicians were forced to enlarge the number system.

> Historical motivation
>
> Cardano's formula showed that even when a polynomial equation has real solutions, intermediate steps in the algebra may require square roots of negative numbers. This is one of the main historical reasons for introducing complex numbers.

Exercise 2

1. Verify directly that $x = -2$ solves $x^3 = 3x - 2$.
2. Verify directly that $x = 3$ solves $x^3 = 6x + 9$.
3. Explain why the expression $\sqrt{-16}$ is a problem if we only allow real numbers.

3 Imaginary Numbers

3.1 The imaginary unit

To handle square roots of negative numbers, we introduce a new symbol:

$$i = \sqrt{-1}.$$

This is called the *imaginary unit*. Its defining property is

$$i^2 = -1.$$

Once this symbol has been introduced, any square root of a negative real number can be rewritten in i-form. If $c > 0$, then

$$\sqrt{-c} = \sqrt{c(-1)} = \sqrt{c}\sqrt{-1} = \sqrt{c}\, i.$$

For example,

$$\sqrt{-121} = \sqrt{121}\sqrt{-1} = 11i,$$
$$\sqrt{-16} = 4i,$$

and

$$\sqrt{-54} = \sqrt{54}\sqrt{-1} = 3\sqrt{6}\,i.$$

Writing negative square roots in i-form

If $c > 0$, then

$$\sqrt{-c} = \sqrt{c}\,i.$$

When writing answers in standard i-form, the i should be written outside the radical.

3.2 Arithmetic with imaginary numbers

Once everything is written in i-form, arithmetic becomes straightforward. For real numbers a and b,

$$ai + bi = (a + b)i,$$

$$ai - bi = (a - b)i,$$

and

$$(ai)(bi) = ab\,i^2 = -ab.$$

So, for instance,

$$\sqrt{-9} + \sqrt{-49} = 3i + 7i = 10i,$$

and

$$\sqrt{-32} - 2\sqrt{-2} = 4\sqrt{2}\,i - 2\sqrt{2}\,i = 2\sqrt{2}\,i.$$

Exercise 3

Write each number in i-form.

1. $\sqrt{-36}$
2. $\sqrt{-\frac{16}{25}}$
3. $\sqrt{-80}$
4. $\frac{\sqrt{-48}}{\sqrt{-3}}$

Exercise 4

Perform each operation.

1. $\sqrt{-4} + \sqrt{-64}$
2. $\sqrt{-18} - \sqrt{-8}$
3. $\sqrt{-16} + \sqrt{-25}$
4. $\sqrt{-75} - 5\sqrt{-3}$

4 Complex Numbers

4.1 Definition

A *complex number* is a number of the form

$$a + bi,$$

where a and b are real numbers. The number a is called the *real part*, and the number bi is called the *imaginary part*. If $b = 0$, then the complex number is just a real number. If $a = 0$ and $b \neq 0$, then the complex number is called *purely imaginary*.

Definition of a complex number

If $a, b \in \mathbb{R}$, then

$$z = a + bi$$

is a complex number. We usually say that this is the *standard form* of a complex number.

The complex numbers are written $\mathbb{C}$. They contain the real numbers $\mathbb{R}$, which in turn contain the rational numbers $\mathbb{Q}$, the integers $\mathbb{Z}$, and the natural numbers $\mathbb{N}$:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

4.2 Equality of complex numbers

Two complex numbers are equal precisely when their real parts are equal and their imaginary parts are equal. In symbols,

$$a + bi = c + di \qquad \Longleftrightarrow \qquad a = c \text{ and } b = d.$$

For example,

$$2^2 + 3i = 4 - (2 - 5)i$$

because $2^2 = 4$ and $3 = -(2 - 5) = 3$.

Likewise,

$$(3 - 2) - i = i + 3 - (4 + 2i) + 2$$

because the left-hand side is

$$1 - i,$$

while the right-hand side is

$$3 + i - 4 - 2i + 2 = 1 - i.$$

But

$$7 - i^2 \neq 6,$$

because

$$7 - i^2 = 7 - (-1) = 8.$$

Exercise 5

Determine whether the following pairs are equal.

1. $1 + 4i$ and $3 - (2 - 4i)$
2. $(4 + i) - (1 - 3i)$ and $3 + 4i$
3. $5 - 2i^2$ and 3

5 Arithmetic of Complex Numbers

5.1 Addition and subtraction

To add or subtract complex numbers, we combine the real parts separately from the imaginary parts:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

This is very similar to combining like terms in algebra.

For example,

$$(3 - i) + (-2 + 4i) = 1 + 3i,$$
$$3i + (5 - 3i) = 5,$$

and

$$4 - (-1 + 5i) + (7 + 2i) = 12 - 3i.$$

5.2 Multiplication

To multiply complex numbers, we expand as usual and use the rule $i^2 = -1$:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2.$$

Since $i^2 = -1$, this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

For example,

$$(2 + 3i)(1 - 4i) = 2 - 8i + 3i - 12i^2 = 14 - 5i.$$

Arithmetic of complex numbers

For complex numbers $a + bi$ and $c + di$,

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Exercise 6

Simplify each expression.

1. $(2+5i)+(-1-3i)$
2. $4i+(7-4i)$
3. $6-(-2+3i)+(1-5i)$
4. $(6+3i)+(2-\sqrt{-8})-\sqrt{-4}$
5. $(1+2i)(3-i)$

6 Complex Conjugates and Division

6.1 Complex conjugates

If

$$z = a + bi,$$

then its *complex conjugate* is

$$\overline{z} = a - bi.$$

This is directly analogous to the radical conjugates we studied last lecture. When a complex number is multiplied by its conjugate, the imaginary parts cancel:

$$(a+bi)(a-bi) = a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 + b^2.$$

So the product of a complex number with its conjugate is always a nonnegative real number.

For example,

$$(3-4i)(3+4i) = 9 + 16 = 25.$$

Product with the conjugate

If $z = a + bi$, then

$$z\overline{z} = (a+bi)(a-bi) = a^2 + b^2.$$

6.2 Division by a complex number

Conjugates allow us to divide by complex numbers. If $c + di \neq 0$, then

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}.$$

The denominator becomes

$$(c+di)(c-di) = c^2 + d^2,$$

which is real. After expansion we obtain

$$\frac{a+bi}{c+di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}.$$

As an example,

$$\frac{4+7i}{2-3i} = \frac{4+7i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{(4+7i)(2+3i)}{2^2+3^2}.$$

Expanding the numerator gives

$$(4+7i)(2+3i) = 8 + 12i + 14i + 21i^2 = -13 + 26i,$$

so

$$\frac{4+7i}{2-3i} = -1 + 2i.$$

Likewise,

$$\frac{1+2i}{1-i} = \frac{(1+2i)(1+i)}{1^2+1^2} = \frac{1+i+2i+2i^2}{2} = \frac{-1+3i}{2} = -\frac{1}{2} + \frac{3}{2}i.$$

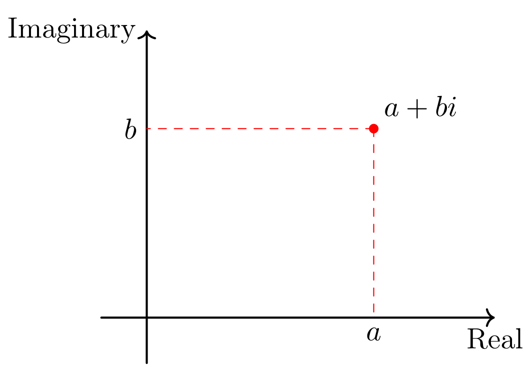
Exercise 7

1. Find the conjugate of $2+5i$, then multiply the number by its conjugate.
2. Simplify $\frac{3+5i}{1-2i}$.
3. Simplify $\frac{2-i}{2+i}$.

7 The Geometry of Complex Numbers

7.1 The Argand plane

A complex number $a + bi$ can be pictured as the point (a, b) in a plane. The horizontal axis records the real part, and the vertical axis records the imaginary part. This picture is called the *Argand plane*.

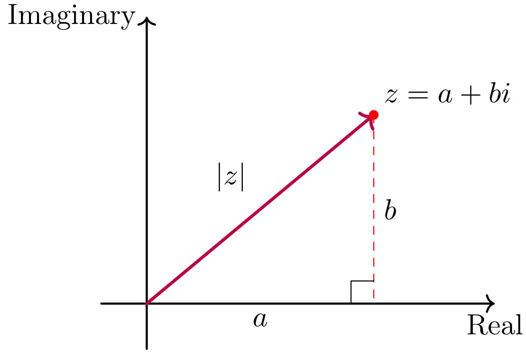


From this point of view, a complex number is not so mysterious. It is simply an ordered pair of real numbers together with a particularly useful multiplication rule.

7.2 Magnitude

If $z = a + bi$, then the distance from the origin to the point (a, b) is called the *magnitude* or *absolute value* of z, written $|z|$. By Pythagoras' theorem,

$$|z| = \sqrt{a^2 + b^2}.$$



For example, if

$$z = 3 - 4i,$$

then

$$|z| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

7.3 A few geometric facts

The complex conjugate

$$a + bi \longmapsto a - bi$$

reflects a point across the real axis. Since this does not change the distance from the origin, conjugation preserves magnitude:

$$|a + bi| = |a - bi|.$$

Also, multiplying a complex number by i rotates it by a quarter-turn in the plane. In particular, multiplying by i, $-i$, or -1 does not change its magnitude.

Finally, because complex numbers can be added and multiplied by real scalars, all of the usual vector language can also be applied to them. In this sense, $\mathbb{C}$ may be viewed as a two-dimensional vector space over $\mathbb{R}$, but with an additional multiplication operation built into it.

Magnitude

If $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2}.$$

Also,

$$z\overline{z} = |z|^2.$$

Exercise 8

Let $z = -1 + 2i$.

1. State the real and imaginary parts of z.
2. Find the complex conjugate of z.
3. Find $|z|$.
4. Compute $z\overline{z}$, and compare your answer with $|z|^2$.

Solutions to the Exercises

Exercise 1

1.
$$\frac{\sqrt{2}}{1-\sqrt{3}} = \frac{\sqrt{2}}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{\sqrt{2}+\sqrt{6}}{1-3} = -\frac{\sqrt{2}+\sqrt{6}}{2}.$$

2.
$$\frac{5}{3-\sqrt{5}} = \frac{5}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{5(3+\sqrt{5})}{9-5} = \frac{15+5\sqrt{5}}{4}.$$

3.
$$\sqrt{3x+1} = 5 \quad \Longrightarrow \quad 3x+1 = 25 \quad \Longrightarrow \quad 3x = 24 \quad \Longrightarrow \quad x = 8.$$

Exercise 2

1.
$$(-2)^3 = -8 \qquad \text{and} \qquad 3(-2) - 2 = -8.$$
So $x = -2$ is a solution.

2.
$$3^3 = 27 \qquad \text{and} \qquad 6(3) + 9 = 18 + 9 = 27.$$
So $x = 3$ is a solution.

3. Over the real numbers, $\sqrt{a}$ only exists when $a \geq 0$. Since $-16 < 0$, the expression $\sqrt{-16}$ is not a real number.

Exercise 3

1.
$$\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i.$$

2.
$$\sqrt{-\frac{16}{25}} = \sqrt{\frac{16}{25}}\sqrt{-1} = \frac{4}{5}i.$$

3.
$$\sqrt{-80} = \sqrt{80}\sqrt{-1} = 4\sqrt{5}\,i.$$

4.
$$\frac{\sqrt{-48}}{\sqrt{-3}} = \frac{4\sqrt{3}\,i}{\sqrt{3}\,i} = 4.$$

Exercise 4

1. $$\sqrt{-4} + \sqrt{-64} = 2i + 8i = 10i.$$

2. $$\sqrt{-18} - \sqrt{-8} = 3\sqrt{2}\,i - 2\sqrt{2}\,i = \sqrt{2}\,i.$$

3. $$\sqrt{-16} + \sqrt{-25} = 4i + 5i = 9i.$$

4. $$\sqrt{-75} - 5\sqrt{-3} = 5\sqrt{3}\,i - 5\sqrt{3}\,i = 0.$$

Exercise 5

1. Yes, because
$$3 - (2 - 4i) = 3 - 2 + 4i = 1 + 4i.$$

2. Yes, because
$$(4 + i) - (1 - 3i) = 4 + i - 1 + 3i = 3 + 4i.$$

3. No, because
$$5 - 2i^2 = 5 - 2(-1) = 7 \neq 3.$$

Exercise 6

1. $$(2 + 5i) + (-1 - 3i) = 1 + 2i.$$

2. $$4i + (7 - 4i) = 7.$$

3. $$6 - (-2 + 3i) + (1 - 5i) = 6 + 2 - 3i + 1 - 5i = 9 - 8i.$$

4. First write the radicals in i-form:
$$\sqrt{-8} = 2\sqrt{2}\,i, \qquad \sqrt{-4} = 2i.$$
Hence
$$(6 + 3i) + (2 - \sqrt{-8}) - \sqrt{-4} = (6 + 3i) + (2 - 2\sqrt{2}\,i) - 2i = 8 + (1 - 2\sqrt{2})i.$$

5. $$(1 + 2i)(3 - i) = 3 - i + 6i - 2i^2 = 5 + 5i.$$

Exercise 7

1. The conjugate of $2+5i$ is $2-5i$, and
$$(2+5i)(2-5i) = 2^2 + 5^2 = 29.$$

2.
$$\frac{3+5i}{1-2i} = \frac{3+5i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{(3+5i)(1+2i)}{1^2+2^2}.$$
Expanding,
$$(3+5i)(1+2i) = 3 + 6i + 5i + 10i^2 = -7 + 11i.$$
So
$$\frac{3+5i}{1-2i} = \frac{-7+11i}{5} = -\frac{7}{5} + \frac{11}{5}i.$$

3.
$$\frac{2-i}{2+i} = \frac{2-i}{2+i} \cdot \frac{2-i}{2-i} = \frac{(2-i)(2-i)}{2^2+1^2}.$$
Expanding,
$$(2-i)(2-i) = 4 - 4i + i^2 = 3 - 4i.$$
Hence
$$\frac{2-i}{2+i} = \frac{3-4i}{5} = \frac{3}{5} - \frac{4}{5}i.$$

Exercise 8

Let $z = -1 + 2i$.

1. The real part is -1, and the imaginary part is $2i$.
2. The complex conjugate is
$$\overline{z} = -1 - 2i.$$
3.
$$|z| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}.$$
4.
$$z\overline{z} = (-1+2i)(-1-2i) = 1 + 4 = 5.$$
Also,
$$|z|^2 = (\sqrt{5})^2 = 5.$$
So indeed
$$z\overline{z} = |z|^2.$$