MAT140: Lecture 19 Handout

*Solutions of Quadratic Equations*

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Last lecture we introduced radicals and complex numbers. Earlier in the course, we also studied factorisation, graphs of equations, and the basic language of functions. In this lecture, we will tie together all of these loose threads. Quadratic equations sit at the centre of several themes in the course because they can be approached in two complementary ways: *algebraically*, by manipulating symbols until the equation reveals its solutions, and *geometrically*, by interpreting those solutions as features of a graph.

Today we will complete our discussion of quadratics from both viewpoints. On the algebraic side, we will review several methods for solving quadratic equations, explain the idea of completing the square, derive the quadratic formula, and use the discriminant to predict what the solutions look like. On the geometric side, we will study the graphs of quadratic functions, locate their vertices and axes of symmetry, and learn how to sketch parabolas from their equations. In this sense, this lecture acts as a small summary of several themes from the course: factorisation, radicals, complex numbers, and the correspondence between algebra and geometry.

Today we will:

1. Review the main algebraic methods for solving quadratic equations.
2. Introduce equations of quadratic form and solve them by substitution.
3. Explain completing the square, both algebraically and geometrically.
4. Derive and apply the quadratic formula, and interpret the discriminant.
5. Study graphs of quadratic functions, including vertex form, axes of symmetry, and basic sketching.
6. Write equations of parabolas from geometric information.

As with the previous handouts, our goal is not merely to memorise formulas. We want to understand where the formulas come from, why different methods work, and what the algebra is actually saying about the geometry of the graph.

## 1 Quadratic Equations and Their Solutions

Recall that a quadratic equation is an equation in which the highest power of the variable is 2. The general form is

$$ax^2 + bx + c = 0,$$

where here $a$, $b$, and $c$ are real numbers and $a \neq 0$. Examples include:

$$x^2 + 5x - 24 = 0, \qquad 3x^2 + 11x - 4 = 0, \qquad x^2 - 2x + 1 = 0.$$

In order to solve a quadratic equation, we want to somehow find all values of $x$ that make the equation true. Depending on the equation, there may be two different solutions, one repeated solution, or no real solutions at all. We will now discuss three initial methods for solving quadratic equations.

## 1.1 Method 1: Factorisation

When a quadratic polynomial can be factorised, the easiest method is often to use the zero-factor property. For instance,

$$x^2 + 5x = 24$$

is first rewritten in general form:

$$x^2 + 5x - 24 = 0.$$

This polynomial factorises as

$$(x + 8)(x - 3) = 0.$$

Therefore,

$$x + 8 = 0 \quad \text{or} \quad x - 3 = 0,$$

so the solutions are $x = -8$ and $x = 3$. Underpinning this method is the so-called *zero factor property*, which we saw in Lecture 10.

> Zero-factor property
>
> If $ab = 0$, then at least one factor must be zero. In other words,
>
> $$ab = 0 \implies a = 0 \text{ or } b = 0.$$

This property only applies when the right-hand side of the equation is equal to zero. So, in order to solve quadratic equations using this method, it is sometimes necessary to first rearrange the equation. To see this in action, consider the equation $3x^2 + 11x = 4$. This can be solved by factorisation, but we first have to rearrange it:

$$3x^2 + 11x - 4 = 0 \qquad \Rightarrow \qquad (3x - 1)(x + 4) = 0,$$

so, the solutions are

$$x = \frac{1}{3} \qquad \text{and} \qquad x = -4.$$

At this stage, it should be noted that factorisation does not always work nicely, some quadratics cannot be factorised with integers, and some do not factor over the real numbers *at all*. So, we occasionally need to use other methods.

**Exercise 1**

Solve each equation.

1. $x^2 + 2x = 15$
2. $2x^2 = 7 - 5x$
3. $x^4 - 10x^2 + 9 = 0$

## 1.2 Method 2: The square root property

A second useful method appears when a quadratic equation can be written in the form

$$(\text{something})^2 = k.$$

In this situation, we can apply the square root property and solve it generally. When discussing square roots in Lecture 17, we emphasized the use of taking the principal root, e.g. we would write $\sqrt{4} = 2$ instead of writing $\sqrt{4} = \pm 2$. However, this is different when solving equations involving variables. If we were to write something like

$$x^2 = 4,$$

then we observe that there are now *two* real numbers that make this statement true, namely $x = 2$ *and* $x = -2$. Therefore, the solutions are $x = \pm\sqrt{4} = \pm 2$.

As an example, consider the equation $3x^2 = 15$. Dividing both sides by 3 gives

$$x^2 = 5 \qquad \Rightarrow \qquad x = \pm\sqrt{5}.$$

This method also connects naturally to the complex numbers of Lecture 18. For example, consider the equation $x^2 + 8 = 0$. Observe that this cannot be factorised, and since it rewrites as $x^2 = -8$, we can see that there are no real solutions to this equation. However, if we rearrange to $x^2 = -8$ and then apply the square root property, we can obtain *complex* solutions to the equation:

$$x^2 + 8 = 0 \quad \Rightarrow \quad x^2 = -8 \quad \Rightarrow \quad x = \pm\sqrt{-8} = \pm\sqrt{8}\,i = \pm 2\sqrt{2}\,i.$$

**Square root property**

Given an algebraic expression $u$, if $u^2 = k$, then

$$u = \pm\sqrt{k}.$$

When $k > 0$, this gives two real solutions. When $k = 0$, it gives one repeated solution. When $k < 0$, the solutions are non-real complex.

It should be noted that this approach works for *algebraic expressions*, rather than simply for variables alone. For example, consider the equation $(x-2)^2 = 10$. We may perform the square root first, giving:

$$x - 2 = \pm\sqrt{10}.$$

Rearranged, this gives the pair of solutions:

$$x = 2 \pm \sqrt{10}.$$

**Exercise 2**

Solve each equation by using the square root property.

1. $5x^2 = 45$
2. $(x+1)^2 = 7$
3. $(3x-6)^2 - 8 = 0$
4. $x^2 + 12 = 0$
5. $(x-4)^2 = -3$

### 1.3 Method 3: Equations of quadratic form

Sometimes an equation is not literally quadratic in $x$, but it becomes quadratic after a substitution. For example,

$$x^4 - 13x^2 + 36 = 0$$

is not a quadratic equation in $x$, but, if we let $u = x^2$ then the equation simplifies to:

$$u^2 - 13u + 36 = 0,$$

which *is* a quadratic equation. This particular equation can be solved for by factorisation:

$$(u-4)(u-9) = 0 \qquad \Rightarrow \qquad u = 4 \quad \text{or} \quad u = 9.$$

Now, we may use the fact that $x^2 = u$ and substitute these solutions back to find the solutions in terms of $x$:

$$x^2 = 4 \quad \Longrightarrow \quad x = \pm 2,$$

and

$$x^2 = 9 \quad \Longrightarrow \quad x = \pm 3.$$

So, our original equation has four solutions:

$$x = -3,\ -2,\ 2,\ 3.$$

This technique works for many equations whose exponents come in a repeated pattern, such as $x^4$ and $x^2$, or $x$ and $\sqrt{x}$, or $x^{2/3}$ and $x^{1/3}$, though in some cases we must check domain restrictions after substituting back.

**Equations of quadratic form via substitution**

If an equation can be rewritten using a substitution

$$u = (\text{some expression in } x),$$

so that it becomes

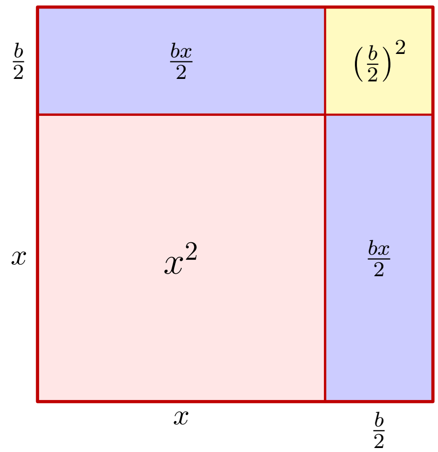
$$au^2 + bu + c = 0,$$

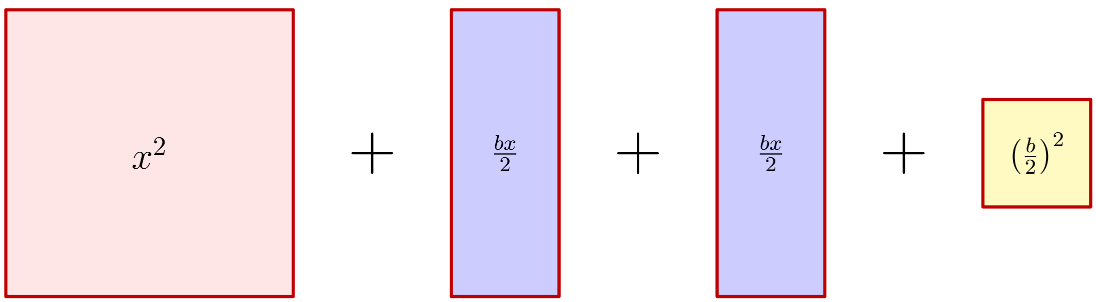
then we can first solve for $u$, and only afterwards return to $x$.

## 2 Completing the Square

### 2.1 The geometric idea

Consider a square of side length $x + \frac{b}{2}$. We can break this length down into a piece of length $x$ and a piece of length $\frac{b}{2}$, so that the total area of the square may be represented as a sum:



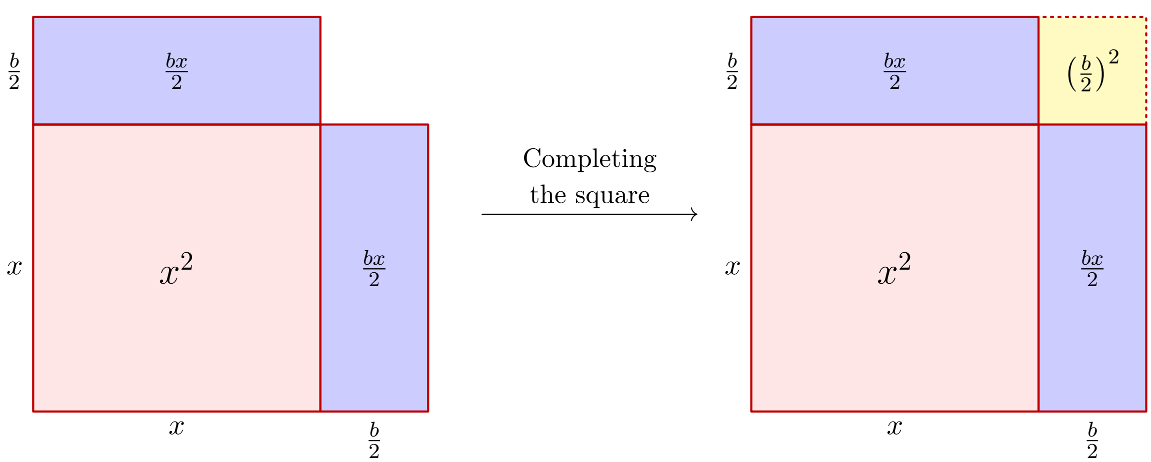

Since the entire area of the square equals $\left(x+\frac{b}{2}\right)^2$, the diagram above can be understood as a geometric realization of the FOIL expansion:

$$\left(x+\frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2.$$

Suppose now that we have the expression:

$$x^2 + bx.$$

We notice that this cannot be cleanly written as a perfect square. However, it is *very close*. If we were to add in the extra term $\left(\frac{b}{2}\right)^2$, then we will obtain an expression that can be written as $\left(x+\frac{b}{2}\right)^2$. This process is known as *completing the square*, since geometrically we have:



Completing the square

To complete the square for the expression

$$x^2 + bx,$$

add the square of half the coefficient of $x$:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

For example, if we want to complete the square for

$$x^2 - 8x,$$

then the coefficient of $x$ is $-8$, so we add $\left(\frac{-8}{2}\right)^2 = 16$ into the expression:

$$x^2 - 8x + 16 = (x - 4)^2.$$

## 2.2 Completing the square to solve equations

Completing the square can also be used to solve quadratic equations. However, when doing so, it is important to remember that completing the square fundamentally changes an expression, since we are adding something onto the expression. There is a risk, therefore, in simply completing the square: if we add something to one side of the equation, then we *must* add the same thing to the other side in order to preserve equality.

Solving by completing the square

To solve a quadratic equation by completing the square:

1. Move the constant term to the other side.
2. If necessary, divide so the coefficient of $x^2$ becomes 1.
3. Add the square of half the coefficient of $x$ to both sides.
4. Rewrite the left-hand side as a square.
5. Use the square root property.

As a first example, consider the equation:

$$x^2 + 12x = 0.$$

Now, upon inspection we know that $x = 0$ or $x = -12$ are both solutions to this equation. However, we will now demonstrate this by completing the square. Here, we observe that $b = 12$, so completing the square gives

$$x^2 + 12x \longrightarrow x^2 + 12x + \left(\frac{12}{2}\right)^2 = x^2 + 12x + 36 = (x + 6)^2.$$

If we would like to use this in the original equation, then we will need to add 36 to *both sides* at the same time:

$$x^2 + 12x + 36 = 36.$$

Using that the left-hand side is now a square, we can take square roots:

$$(x+6)^2 = 36 \quad \Rightarrow \quad x+6 = \pm\sqrt{36} = \pm 6.$$

Therefore $x = -6 \pm 6$, which gives two solutions: $x = 0$ and $x = -12$ as expected.

As a more complicated example, consider the equation $2x^2 - x - 2 = 0$. To solve this, we first move the constant term across to the other side and then divide by 2:

$$2x^2 - x = 2 \quad \Rightarrow \quad x^2 - \frac{1}{2}x = 1.$$

Here, we recognise that the left-hand side of this equation can be completed into a square, here $b = -\frac{1}{2}$. So, we complete the square by adding

$$\left(\frac{-1/2}{2}\right)^2 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16}$$

to both sides:

$$x^2 - \frac{1}{2}x + \frac{1}{16} = 1 + \frac{1}{16}.$$

Once we have done this, we may use the equality $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$ with $b = -\frac{1}{2}$ to get:

$$\left(x - \frac{1}{4}\right)^2 = \frac{17}{16}.$$

Taking square roots of both sides gives

$$x - \frac{1}{4} = \pm\frac{\sqrt{17}}{4},$$

and therefore:

$$x = \frac{1}{4} \pm \frac{\sqrt{17}}{4}.$$

**Exercise 3**

Rewrite each expression in completed-square form.

1. $x^2 + 4x$
2. $x^2 + 12x$
3. $x^2 - 6x$
4. $2x^2 - 12x$

**Exercise 4**

Solve each equation by completing the square.

1. $x^2 - 8x + 3 = 0$
2. $3x^2 + 6x - 9 = 0$

# 3 The Quadratic Formula and the Discriminant

## 3.1 Deriving the quadratic formula

In the previous example, we derived a solution to $2x^2 - x - 2 = 0$ by rearranging the equation so that the left-hand side was of the form $x^2 + bx$, so that we could complete the square. In a sense, we can try this approach with a general quadratic also. Suppose that we have an equation of the form

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. From here, we can subtract $c$ from both sides and then divide by $a$ to get:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

From here, we may complete the square for the expression on the left-hand side to get:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$$

Using the fact that the left-hand side is now a square, we may factorise and then take square roots:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \Longrightarrow \qquad x + \frac{b}{2a} = \pm\sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2}. \qquad (*)$$

The expression inside the square root can be simplified using exponent rules and by adding these two rational expressions directly:

$$-\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

When we take the square root of this expression, we will have:

$$\sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Therefore, our equation $(*)$ simplifies to:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The equality above gives us a general prescription for solving a quadratic: we simply take the values of $a, b$ and $c$ and plug them into the formula, and the result will describe the solutions of the original quadratic equation. This is, of course, the famous *quadratic formula*.

**The quadratic formula**

If

$$ax^2 + bx + c = 0, \qquad a \neq 0,$$

then its solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula is the most reliable general method because it applies to every quadratic equation, producing real or complex solutions as appropriate.

As an example, consider the equation:

$$x^2 + 6x = 16.$$

In general form, this becomes

$$x^2 + 6x - 16 = 0 \qquad \Longrightarrow \qquad a = 1, b = 6, \text{ and } c = -16.$$

Using the quadratic formula, we have:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(-16)}}{2(1)} = \frac{-6 \pm \sqrt{36 + 64}}{2} = \frac{-6 \pm 10}{2}.$$

Therefore,

$$x = 2 \qquad \text{or} \qquad x = -8.$$

**Exercise 5**

Use the quadratic formula to solve

$$x^2 + 4x = 5.$$

## 3.2 The discriminant

Inside the quadratic formula sits the quantity

$$b^2 - 4ac.$$

This is called the *discriminant*. It is important, because it tells us what kind of solutions the quadratic equation has.

The discriminant

For a quadratic equation

$$ax^2 + bx + c = 0,$$

the discriminant is

$$\Delta = b^2 - 4ac.$$

Its sign determines the nature of the solutions. When $a$, $b$, and $c$ are rational numbers, we can say more precisely:

1. If $\Delta$ is a positive perfect square, there are two distinct rational solutions.
2. If $\Delta > 0$ but is not a perfect square, there are two distinct irrational real solutions.
3. If $\Delta = 0$, there is one repeated real solution.
4. If $\Delta < 0$, there are two distinct imaginary solutions.

As an example of discriminants, we will determine the nature of the solutions of three quadratic equations.

- The equation $x^2 - x + 2 = 0$has $a = 1, b = -1$ and $c = 2$, so its discriminant is:

$$(-1)^2 - 4(1)(2) = 1 - 8 = -7 < 0.$$

  Here the discriminant is less than zero, so this equation has two non-real complex solutions.
- The equation $2x^2 - 3x - 2 = 0$ has $a = 2, b = -3$ and $c = -2$, so its discriminant is:

$$(-3)^2 - 4(2)(-2) = 9 + 16 = 25.$$

  This is a positive perfect square, so it has two distinct rational solutions.
- The equation $x^2 - 2x + 1 = 0$ has $a = 1, b = -2$ and $c = 1$, so its discriminant is:

$$(-2)^2 - 4(1)(1) = 4 - 4.$$

  This is equal to zero, so the equation has one repeated real solution.

**Exercise 6**

Use the discriminant to determine the type of solution(s) of each equation.

1. $x^2 + 4x + 7 = 0$
2. $3x^2 - x - 2 = 0$
3. $x^2 + 6x + 9 = 0$
4. $x^2 - 2x - 1 = 9$

# 4 Graphs of Quadratic Functions

## 4.1 Parabolas, vertex form, and symmetry

A quadratic *function* has the form

$$f(x) = ax^2 + bx + c, \qquad a \neq 0.$$

Its graph is called a *parabola*. The most useful form for graphing is the completed-square form

$$f(x) = a(x - h)^2 + k.$$

This is often called the *vertex* form. From this form, we can read off the key geometric information immediately.

**Vertex form of a parabola**

Given

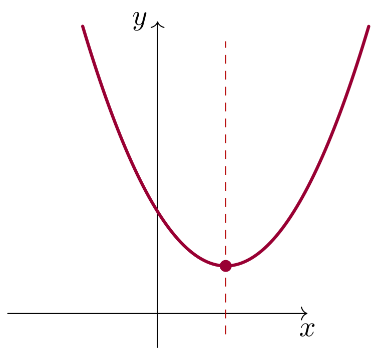
$$f(x) = a(x - h)^2 + k,$$

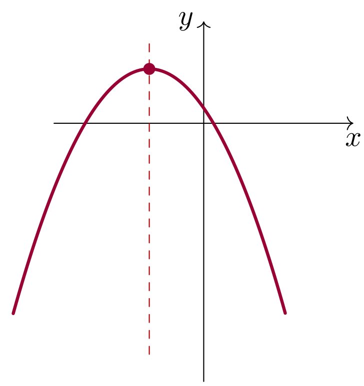
a quadratic function with $a \neq 0$:

- the graph is a parabola,
- the vertex is $(h, k)$,
- the axis of symmetry is $x = h$.

Because $(x - h)^2 \geq 0$, the $y$-coordinate $k$ of the vertex is the minimum value when $a > 0$, and the maximum value when $a < 0$.

Every parabola is symmetric about its axis. This means that if the graph were folded along the axis of symmetry, the two halves would match.




### 4.2 Finding the vertex by completing the square

If we have a quadratic function that is not presented in vertex form, then we can complete the square to find its vertex. For example, consider the function

$$f(x) = x^2 - 6x + 5.$$

Completing the square on the first two terms gives:

$$f(x) = x^2 - 6x + 9 - 9 + 5 = (x - 3)^2 - 4.$$

So the vertex is

$$(3, -4).$$

Since the coefficient of the squared term is positive, the parabola opens upward, and therefore $-4$ is the minimum value of the function.

### 4.3 A general vertex formula

Generally speaking, we can also complete the square to find the vertex of a function of the form:

$$f(x) = ax^2 + bx + c.$$

If we factor out $a$ from the first two terms, we get

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c.$$

Completing the square inside the brackets gives:

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}.$$

So

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.$$

From this form we can read off that the axis of symmetry is

$$x = -\frac{b}{2a}.$$

The axis formula

For the quadratic function

$$f(x) = ax^2 + bx + c, \qquad a \neq 0,$$

the axis of symmetry is

$$x = -\frac{b}{2a}.$$

This is often the quickest way to locate the $x$-coordinate of the vertex.

## 4.4 Sketching a parabola

To sketch a parabola, the most efficient strategy is usually:

1. determine the vertex and axis of symmetry, either by completing the square or by using $x = -\frac{b}{2a}$;
2. plot the vertex, the axis, the $y$-intercept, and any $x$-intercepts if they exist;
3. use symmetry to obtain a few extra points;
4. use the sign of $a$ to decide whether the graph opens upward or downward.

Consider

$$y = x^2 + 6x + 8.$$

Completing the square gives

$$y = x^2 + 6x + 9 - 9 + 8 = (x+3)^2 - 1.$$

So the vertex is

$$(-3, -1),$$

and the axis is

$$x = -3.$$

The $y$-intercept is found by setting $x = 0$:

$$y = 8,$$

so the graph passes through $(0, 8)$.

The $x$-intercepts are found by solving

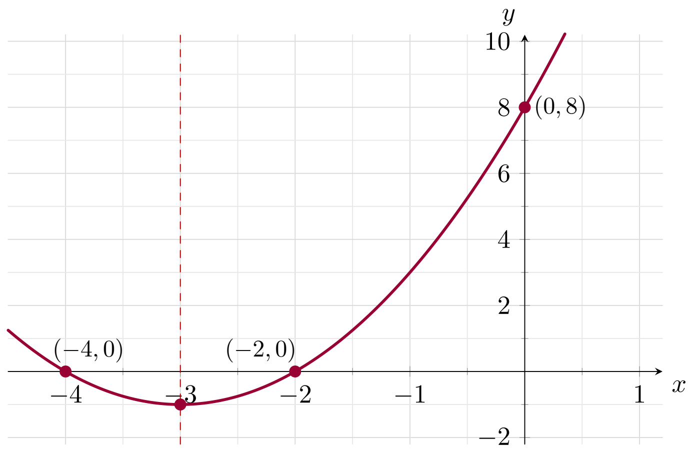
$$x^2 + 6x + 8 = 0 = (x+4)(x+2).$$

Hence

$$x = -4 \qquad \text{or} \qquad x = -2,$$

so the graph crosses the $x$-axis at

$$(-4, 0) \qquad \text{and} \qquad (-2, 0).$$




Since the leading coefficient is positive, the parabola opens upward. In fact, this graph can also be seen as the graph of $y = x^2$ shifted three units to the left and one unit downward.

**Exercise 7**

For the quadratic function

$$f(x) = x^2 + 4x + 1,$$

find:

1. the vertex,
2. the axis of symmetry,
3. the minimum or maximum value of the function.

**Exercise 8**

Sketch the parabola

$$y = x^2 - 4x + 3.$$

Find its vertex, axis of symmetry, $x$-intercepts, and $y$-intercept.

## 5 Writing the Equation of a Parabola

If a parabola has a vertical axis and vertex $(h, k)$, then its equation has the form

$$y = a(x - h)^2 + k.$$

The only unknown is the parameter $a$, which tells us how the parabola stretches and whether it opens upward or downward.

Suppose we are told that the parabola has vertex $(-2, 1)$ and $y$-intercept $(0, -3)$. Since the vertex is $(h, k) = (-2, 1)$, the equation must be

$$y = a(x + 2)^2 + 1.$$

Now use the fact that $(0, -3)$ lies on the graph:

$$-3 = a(0+2)^2 + 1.$$

So

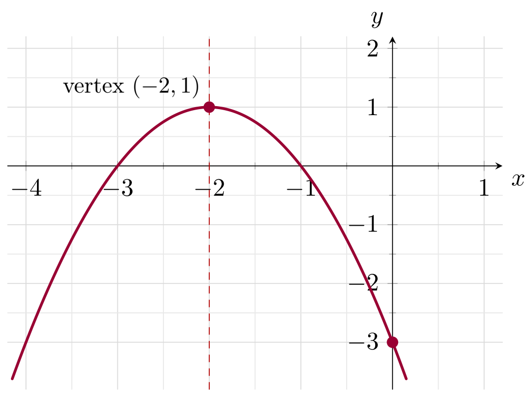
$$-3 = 4a + 1,$$

which implies

$$-4 = 4a \qquad \Longrightarrow \qquad a = -1.$$

Therefore the equation is

$$y = -(x+2)^2 + 1.$$




**Writing an equation from the vertex**

If the vertex of a parabola is $(h, k)$, then start with

$$y = a(x-h)^2 + k.$$

Then substitute the coordinates of one more known point to determine the value of $a$.

**Exercise 9**

Write the equation of the parabola with vertex $(1, -2)$ and $y$-intercept $(0, 1)$.

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## Solutions to the Exercises

### Exercise 1

1. $$x^2 + 2x = 15 \quad \Longrightarrow \quad x^2 + 2x - 15 = 0 \quad \Longrightarrow \quad (x+5)(x-3) = 0.$$

   Hence

   $$x = -5 \qquad \text{or} \qquad x = 3.$$

2.

$$2x^2 = 7 - 5x \quad \Longrightarrow \quad 2x^2 + 5x - 7 = 0 \quad \Longrightarrow \quad (2x+7)(x-1) = 0.$$

Hence

$$x = -\frac{7}{2} \qquad \text{or} \qquad x = 1.$$

3. Let $u = x^2$. Then

$$x^4 - 10x^2 + 9 = 0 \quad \Longrightarrow \quad u^2 - 10u + 9 = 0 \quad \Longrightarrow \quad (u-1)(u-9) = 0.$$

So $u = 1$ or $u = 9$. Therefore

$$x^2 = 1 \Longrightarrow x = \pm 1, \qquad x^2 = 9 \Longrightarrow x = \pm 3.$$

The solutions are

$$x = -3,\ -1,\ 1,\ 3.$$

## Exercise 2

1.

$$5x^2 = 45 \quad \Longrightarrow \quad x^2 = 9 \quad \Longrightarrow \quad x = \pm 3.$$

2.

$$(x+1)^2 = 7 \quad \Longrightarrow \quad x + 1 = \pm\sqrt{7} \quad \Longrightarrow \quad x = -1 \pm \sqrt{7}.$$

3.

$$(3x-6)^2 - 8 = 0 \quad \Longrightarrow \quad (3x-6)^2 = 8.$$

Hence

$$3x - 6 = \pm\sqrt{8} = \pm 2\sqrt{2},$$

so

$$3x = 6 \pm 2\sqrt{2} \quad \Longrightarrow \quad x = 2 \pm \frac{2\sqrt{2}}{3}.$$

4.

$$x^2 + 12 = 0 \quad \Longrightarrow \quad x^2 = -12 \quad \Longrightarrow \quad x = \pm\sqrt{-12} = \pm 2\sqrt{3}\,i.$$

5.

$$(x-4)^2 = -3 \quad \Longrightarrow \quad x - 4 = \pm\sqrt{-3} = \pm\sqrt{3}\,i.$$

Hence

$$x = 4 \pm \sqrt{3}\,i.$$

## Exercise 3

1.

$$x^2 + 4x = x^2 + 4x + 4 - 4 = (x+2)^2 - 4.$$

2.

$$x^2 + 12x = x^2 + 12x + 36 - 36 = (x+6)^2 - 36.$$

3.

$$x^2 - 6x = x^2 - 6x + 9 - 9 = (x-3)^2 - 9.$$

4. First factor out the 2:

$$2x^2 - 12x = 2(x^2 - 6x) = 2\big((x-3)^2 - 9\big) = 2(x-3)^2 - 18.$$

### Exercise 4

1. $$x^2 - 8x + 3 = 0 \quad \Longrightarrow \quad x^2 - 8x = -3.$$

   Add 16 to both sides:
   $$x^2 - 8x + 16 = 13.$$
   So
   $$(x-4)^2 = 13,$$
   and therefore
   $$x - 4 = \pm\sqrt{13}.$$
   Hence
   $$x = 4 \pm \sqrt{13}.$$

2. $$3x^2 + 6x - 9 = 0 \quad \Longrightarrow \quad 3x^2 + 6x = 9 \quad \Longrightarrow \quad x^2 + 2x = 3.$$

   Add 1 to both sides:
   $$x^2 + 2x + 1 = 4.$$
   So
   $$(x+1)^2 = 4,$$
   and therefore
   $$x + 1 = \pm 2.$$
   Hence
   $$x = 1 \qquad \text{or} \qquad x = -3.$$

### Exercise 5

$$x^2 + 4x = 5 \quad \Longrightarrow \quad x^2 + 4x - 5 = 0.$$

Using the quadratic formula with $a = 1$, $b = 4$, and $c = -5$,

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-5)}}{2} = \frac{-4 \pm \sqrt{16+20}}{2} = \frac{-4 \pm 6}{2}.$$

Therefore

$$x = 1 \qquad \text{or} \qquad x = -5.$$

### Exercise 6

1. For $x^2 + 4x + 7 = 0$,
   $$\Delta = 4^2 - 4(1)(7) = 16 - 28 = -12.$$
   Since $\Delta < 0$, there are two distinct non-real complex solutions.
2. For $3x^2 - x - 2 = 0$,
   $$\Delta = (-1)^2 - 4(3)(-2) = 1 + 24 = 25.$$
   Since 25 is a positive perfect square, there are two distinct rational solutions.

3. For $x^2 + 6x + 9 = 0$,
$$\Delta = 6^2 - 4(1)(9) = 36 - 36 = 0.$$
So there is one repeated real solution.
4. First rewrite
$$x^2 - 2x - 1 = 9$$
as
$$x^2 - 2x - 10 = 0.$$
Then
$$\Delta = (-2)^2 - 4(1)(-10) = 4 + 40 = 44.$$
Since $44 > 0$ but is not a perfect square, there are two distinct irrational real solutions.

### Exercise 7

We complete the square:
$$f(x) = x^2 + 4x + 1 = (x + 2)^2 - 3.$$
So:

1. the vertex is $(-2, -3)$,
2. the axis of symmetry is $x = -2$,
3. since the parabola opens upward, the minimum value is $-3$.

### Exercise 8

We rewrite
$$y = x^2 - 4x + 3$$
in completed-square form:
$$y = x^2 - 4x + 4 - 4 + 3 = (x - 2)^2 - 1.$$
Therefore:

1. the vertex is $(2, -1)$,
2. the axis of symmetry is $x = 2$,
3. the $y$-intercept is $(0, 3)$,
4. the $x$-intercepts satisfy
$$x^2 - 4x + 3 = 0 = (x - 1)(x - 3),$$
so the intercepts are $(1, 0)$ and $(3, 0)$.

Since the leading coefficient is positive, the parabola opens upward.

## Exercise 9

Start from vertex form:

$$y = a(x - h)^2 + k.$$

With vertex $(1, -2)$, this becomes

$$y = a(x - 1)^2 - 2.$$

Now use the point $(0, 1)$:

$$1 = a(0 - 1)^2 - 2 = a - 2.$$

So

$$3 = a.$$

Hence the equation is

$$y = 3(x - 1)^2 - 2.$$