MAT140 — Lecture 2 Handout
Algebraic Expressions

Mathematics as a Language

When learning a new language, you typically learn:

- **New symbols (letters).** For example, English uses letters like $a, b, c, d, e, \ldots$. Other languages use different symbol systems.
- **Rules for combining symbols.** A valid word in one language may not be a valid word in another, even if it uses similar letters.

A key point is that symbols by themselves are just marks; they only *take meaning* when they are combined correctly and used to refer to some existing concept.

Mathematics is similar. It has *symbols*, *expressions*, *equations/identities*, and *grammar rules* that tell us how to combine things correctly.

Language	**Mathematics**
Alphabet	Symbols
Words	Expressions
Sentences	Equations / identities
Grammar	Rules (properties)

In particular, the algebra that we will see in this course follows this analogy. Throughout this course you will see:

- new symbols,
- new rules for their manipulation, and
- ways to interpret the meaning of all of these new things.

Throughout this lecture, we will start with the very basics of our language-building: we will introduce some new symbols and make clear how they all relate to each other.

1 Evaluating Algebraic Expressions

1.1 What is an algebraic expression?

Algebraic Expression

An **algebraic expression** is a collection of letters (variables) and real numbers (constants) combined using arithmetic operations such as addition, subtraction, multiplication, or division.

Algebraic expressions are among the most basic things you can write down using algebra — they are like *words or phrases* in our new language. Generally, an algebraic expression is made up of the ingredients:

- **Real numbers** (constants), e.g. $1, -7, \frac{1}{2}, 0, \pi, \ldots$
- **Variables**, e.g. $x, y, z, \ldots$ (numbers that are allowed to vary)
- **Arithmetic operations**, e.g. $+, -, \times, \div$ (ways to connect things together)
- **Parentheses:** brackets (and).

The first three collections of symbols are the mathematical content of our language, and the parentheses are there to tell us how to order everything. In a way, the most interesting part of an algebraic expression is the operations: these operations encode the recipe for how to combine variables and constants together.[1]

Examples of algebraic expressions include:

$$x^2 - 4x + 5, \qquad 1 + 4x, \qquad 7y - 3x.$$

Non-examples could be meaningless things like:

$$x - + + \times 3, \qquad 2^{\div}, \qquad 4^{x}_{(+} - - \div .$$

Obviously these make no sense, and they are not algebraic expressions.

1.2 Terms and coefficients

Terms

The **terms** of an algebraic expression are the parts of the expression that are separated by addition.

In practice, subtraction is treated as "adding a negative piece," so the terms are the pieces of the expression you see after rewriting everything using +s. For example, consider the expression

$$x^2 - 4x + 5.$$

Here, the terms are x^2, $-4x$, and 5, i.e. $4x$ is *not* a term. To correctly identify the terms of this expression, it's useful to rewrite it as:

$$x^2 - 4x + 5 \;=\; x^2 + (-4x) + 5.$$

As you can see, we need to separate everything by + symbols in order to correctly identify the terms.

An important point here is that the terms of an algebraic expression really depend on *how it's written*, rather than *what it means*. For instance, the two expressions:

$$x - 1 - 2 \qquad \text{and} \qquad x - 3$$

might *feel* like the same expression, since they are saying the same thing. However, they are actually viewed as two different algebraic expressions, since the first has three terms and the second has only two terms.

[1] As you will see later in the course, even with just a few basic arithmetic operations we can create a rich and interesting collection of algebraic structures. Our approach to studying this new "language of algebra" will be slow: we will start with the simplest examples and concepts first, and then slowly walk to the mountaintop.

Example Problem: identifying terms

Identify the terms of each algebraic expression.

1. $x + 2$ — **A:** x and 2
2. $x + \frac{1}{2}$ — **A:** x and $\frac{1}{2}$
3. $2y - 5x - 7$ — **A:** $2y$, $-5x$, and -7
4. $5(x - 3) + 3x - 4$ — **A:** $5(x - 3)$, $3x$, and -4
5. $4 - 6x + \dfrac{x+9}{3}$ — **A:** 4, $-6x$, and $\dfrac{x+9}{3}$

Now that we have introduced terms, we will talk about another important feature of algebraic expressions: coefficients.

Coefficients

Given a term of an algebraic expression, its **coefficient** is the numerical factor in front of the variable.

In simpler terms, the coefficient of a term is just the number in front of the letters.[2] In the situation where there are no variable letters in the term, the coefficient is simply the number itself. For example, in the algebraic expression $x^2 - 4x + 5$, the coefficients are the numbers 1, -4, and 5.

Example Problem: identifying coefficients

Identify the coefficient of each term.

1. $4x$ — **A:** 4
2. $15x^2$ — **A:** 15
3. $3y$ — **A:** 3
4. 3 — **A:** 3

Summary

- Terms = things separated by addition.
- Coefficients = numbers in front of the terms.

The terms of an algebraic expression are always separated by addition, so if you have a minus sign in front then you need to include that in the term. Terms also depend on the way the expression is written: rewriting an expression using different symbols can change its terms.

1.3 Exponential form

Alongside the four basic arithmetic operations of $+$, $-$, $\times$ and $\div$, later in the course we will also care a lot about exponentiation. This operation is crucial for defining interesting algebraic expressions

[2] The word "coefficient" seems a bit confusing. But, it makes sense once you break it down and look at the deeper meaning. The prefix "co" means "together", like in the word "coworker", which means "someone who you work together with", or the word "coauthor", which means "someone who you write together with". The word "efficient" comes from Latin and means "accomplishing", so the word "coefficient" means something like "coming together to produce a result".

such as polynomials.

Exponential notation

For a real number a and a positive integer n, the expression a^n means

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}.$$

Here a is called the **base** and n is called the **exponent**.

The exponential notation is just a useful notation that represents a repeated action of multiplication. In fact, you have already seen this idea before: multiplication is *itself* just a useful notation that summarises a repeated action of addition:

$$n \cdot a = \underbrace{a + a + a \cdots a}_{n \text{ terms}}.$$

Exponentials like a^n are doing a similar thing. For example:

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81 \qquad 5^1 = 5 \qquad (-2)^3 = (-2) \cdot (-2) \cdot (-2) = -8$$

It is quite common for students to misinterpret the order of operations in algebra. The ordering is:

$$\text{Brackets} \longrightarrow \text{Exponents} \longrightarrow \text{Multiplication/Division} \longrightarrow \text{Addition/Subtraction}$$

In other words: if we have an algebraic expression in which there are multiple operations all mixed up together, then there is an order of priority in how to apply the operations. We do the things in brackets first, then the exponents, then the multiplication or division, and then finally the addition or subtraction. For example:

$$-3^2 = -(3^2) = -(3 \cdot 3) = -9 \qquad \text{but} \qquad (-3)^2 = (-3) \cdot (-3) = 9.$$

Similarly,

$$3x^2 = 3(x^2), \qquad -3x^2 = -(3x^2), \qquad (-3x)^2 = 9x^2.$$

1.4 Evaluating an expression

An algebraic expression often has variables such as x or y. These letters are special, in that they are allowed to potentially take on many different values: x could be 3 or π or -4 if we wanted it to. Including variables in an algebraic expression makes our expression much more *general* — it allows us to talk about *many numbers at once*, instead of committing to fixed numbers like $1, 2, 3, \cdots$. That being said, sometimes we would like to talk about specific numbers instead of these more-general variables. In this case, we commit to a specific value of the variables and compute what the expression is saying in this specific case. This process is often called **evaluating an expression**.

Evaluating an expression

To **evaluate** an algebraic expression, you substitute numerical values for the variables and then simplify using the usual arithmetic operations.

As an example, we will evaluate the expression $4x + 7y - 2z$ for $x = 2$, $y = 3$, and $z = 4$. In order to do this, we simply take these three values and plug them into the expression. This gives us a number that we can calculate:

$$4x + 7y - 2z = 4(2) + 7(3) - 2(4) = 8 + 21 - 8 = 21.$$

In words, we say that "the expression $4x + 7y - 2z$ evaluated at $x = 2$, $y = 3$, and $z = 4$ is 21".

Exercise 1.1

Evaluate each expression using the given values.

1. $\dfrac{x+y}{2}$ for $x = 3$, $y = 7$.
2. $5x^2 + 2y$ for $x = -2$, $y = 4$.
3. $\dfrac{x^2 - y^2}{x - y}$ for $x = 5$, $y = 2$.

2 Simplifying algebraic expressions

Algebraic expressions are a symbolic feature of language: they are just a collection of symbols arranged in some way. Often, two algebraic expressions have the same *meaning*, in that they can be turned into each other using a series of simple rules. In practice, what we really care about is whether or not some complicated-looking expression can be written in a simpler form without changing its meaning — this is what we call **simplifying an expression**. In our language analogy, simplifying an expression means "rewriting a sentence" using grammar rules while keeping the meaning the same.

2.1 Properties of Real Numbers

The real numbers have a collection of useful properties, which are summarised as follows.

Properties of Real Numbers

For real numbers a, b, c (and $a \neq 0$ where indicated):

1. **Commutative property of addition:** $a + b = b + a$.
2. **Commutative property of multiplication:** $ab = ba$.
3. **Associative property of addition:** $(a + b) + c = a + (b + c)$.
4. **Associative property of multiplication:** $(ab)c = a(bc)$.
5. **Additive identity property:** $a + 0 = a$.
6. **Multiplicative identity property:** $a \cdot 1 = a$.
7. **Additive inverse property:** $a + (-a) = 0$.
8. **Multiplicative inverse property:** $a \cdot \frac{1}{a} = 1$ (for $a \neq 0$).
9. **Distributive property:** $a(b + c) = ab + ac$.

All of the above properties are stated generally, so they may appear to be weird and unfamiliar.

However, by picking a few numbers you can see that any of these properties is clearly true.[3] For instance, if we choose property (5) and pick $a = 17$, we can see very clearly that $17 + 0 = 17$.

The rules above can be used to simplify algebraic expressions. We will focus mostly on Property (9), since that is the most interesting one. There are two key techniques:

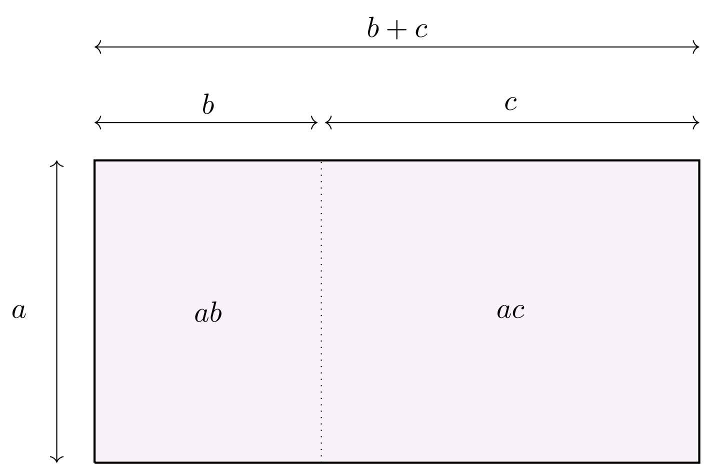
1. simplifying by using distributivity to remove brackets, and
2. simplifying by collecting like terms.

As you will see, these two techniques are really just applications of Property (9).

2.2 Using Distributivity

Before getting to the simplification of algebraic expressions, we will briefly justify the distributive property. Of all of the 9 properties listed above, it is perhaps the least obvious.

In order to show that $a(b + c) = ab + ac$ generally holds, we will interpret numbers as lengths, so that addition means putting two lengths next to each other, and multiplication means the creation of areas. For this geometric picture, assume $a, b, c > 0$.[4] Consider the following diagram.



In this picture, we have combined the areas ab and ac into a single larger rectangle, which is drawn in the thicker lines. Since the length of the total rectangle is $b + c$, and the area of any rectangle is simply base times height, we see visually that $a(b + c)$ has the same area as ab and ac added together. Writing this algebraically, we recover the distributive property: $a(b + c) = ab + ac$.

Now that we believe the distributive property holds, we can use it to manipulate algebraic expressions.

[3] Interestingly, the real number line is not the only mathematical structure that has these 9 properties — the general concept is that of a "field". We will see another example of a "field" later in the course.

[4] The zero cases are simpler. For example, $0(b + c) = 0$ and also $0 \cdot b + 0 \cdot c = 0$, so the rule still holds. Similarly, $a(b + 0) = ab$ and $ab + a \cdot 0 = ab$.

Expanding with Distributivity

To **expand** an expression means to remove parentheses by applying the distributive property.

As an example, we can expand the expression $5(x+7)$ by applying the distributive property:

$$5(x+7) = 5x + 35.$$

2.3 Collecting like terms

Like Terms

Two terms in an expression are called **like terms** if they have the same variable part, i.e. the same variables with the same exponents.

The benefit of like terms is that they can always be collected and simplified. For example, the expression $-3x + x$ can be simplified by collecting these two terms together:

$$-3x + x = -2x.$$

In a sense we can see "collecting like terms" as the opposite of the distributive property. For example, the two like terms $3x^2 + 4x^2$ can be collected by undoing the distributive property:

$$3x^2 + 4x^2 = (3+4)x^2 = 7x^2.$$

Exercise 2.1

Simplify the following expressions by collecting like terms.

1. $6a^2 + 3a^2$
2. $4x - x + 3x$
3. $10 - 3$
4. $5y + 3x - 2y$

Exercise 2.2

Simplify each expression.

1. $3(x+2) + 7$
2. $4(3y-1) - 2y$
3. $-5(2a+3) - 4a$
4. $2(m-4) + 3(m+5)$

2.4 A Geometry Example

Suppose a rectangle has length $l = 4x$ and width $w = 7$. If its perimeter is $P = 54$, find x.

Solution: Recall that the perimeter of a rectangle is

$$P = 2l + 2w.$$

Substitute $l = 4x$, $w = 7$, and $P = 54$:

$$2(4x) + 2(7) = 54 \quad \Rightarrow \quad 8x + 14 = 54 \quad \Rightarrow \quad 8x = 40 \quad \Rightarrow \quad x = 5.$$

3 Basics of equations

3.1 What is an equation?

In our language analogy, algebraic expressions are like expressions/phrases, and equations are like *statements* — we take expressions and make claims.

> Equations
>
> An **equation** is a statement that claims two algebraic expressions are equal.

Simply put, an equation inserts the symbol "=" to connect two expressions together. Examples include:

$$x + 2 = 5, \qquad 3x - 7 = 11, \qquad 4x + 7y - 2z = 21.$$

The equals sign should be interpreted as the word "is". So, an equation is saying something like "the expression on the left and the expression on the right are the same thing". Since this is a claim, it can either be true or false — an equation is a statement that can be *true* or *false* depending on the values of its variables. For example, the equation $x - 1 = 0$ would be true if $x = 1$ but false if $x = 100$. This brings us nicely to the idea of *solving an equation*.

> Solving an Equation
>
> To solve an equation means to find the correct values of the variables that would make the statement true.

Using the same example as before, we would call $x = 1$ a solution to the equation $x - 1 = 0$, since evaluating at $x = 1$ yields the true statement that $0 = 0$. In contrast, the value $x = 100$ is *not* a solution to the equation $x - 1 = 0$, since evaluating at $x = 100$ gives the *false* statement $99 = 0$.

Exercise 3.1

Determine whether or not $x = -2$ and $x = 2$ are solutions of $x^2 - 5 = 4x + 7$.

3.2 The Balancing Idea

In general, we can take an algebraic expression and *do something* to it by applying arithmetic operations. If we have an equation, we need to make sure that whatever we do to the left-hand side of the equation, we must also do to the right-hand side. Doing something changes an expression, and we want to make sure that our two expressions remain equal to each other. As an analogy: if I have two twins that look identical, and one of them gets a tattoo on their arm, then they no longer look identical. The only way that they could remain identical would be if they both got the exact same tattoo. That way, they would still look exactly the same.

To see this principle in another way, suppose that we have two men standing on either side of a seesaw. If both sides have equal weight, the seesaw is level. Suppose that we know the right-hand man has a weight of 75kg, but we only know that the left-hand man has a weight of $(x-10)$kg. If they are both standing on a level seesaw, then the two weights must be equal, i.e. $x-10=75$. If we wanted to figure out the exact weight of the left-hand man, then we need to somehow "undo" the -10kg to see what the value of x is. But, we cannot simply add $+10$kg to the man on the left, because then he would weigh more than 75kg and the seesaw would tip over. Instead, to keep the seesaw level we have to give $+10$kg to *both men at the same time*:

$$x-10+10=75+10 \quad \Rightarrow \quad x=85.$$

3.3 Equivalent Equations

Equivalent equations

Two equations are called **equivalent** if they have the same solution set. When solving, we transform an equation into simpler equivalent equations until the variable is isolated.

There are several techniques that we can use in order to form equivalent equations.

1. **Simplify either side:** remove grouping symbols, combine like terms, or simplify fractions on one or both sides.
$$3x-x=8 \quad \Rightarrow \quad 2x=8.$$
2. **Addition property of equality:** add (or subtract) the same quantity to (from) each side.
$$x-2=5 \quad \Rightarrow \quad x-2+2=5+2 \quad \Rightarrow \quad x=7.$$
3. **Multiplication property of equality:** multiply (or divide) each side by the same *nonzero* quantity.
$$3x=9 \quad \Rightarrow \quad \frac{3x}{3}=\frac{9}{3} \quad \Rightarrow \quad x=3.$$
4. **Interchange the two sides of the equation:**
$$7=x \quad \Rightarrow \quad x=7.$$

Properties (2) and (3) embody the statement "whatever you do to the left, you must also do to the right".

Generally, we want to solve an equation by making a series of these moves to turn a complicated expression into a simpler one, so that we can get an expression that clearly describes what the variable's value should be. These equivalent moves are called "steps" of the solution. We will talk about this in detail in the next lecture. However, for now, here is a simple example:

$$x-5=1 \quad \Rightarrow \quad x-5+5=1+5 \quad \Rightarrow \quad x+0=6 \quad \Rightarrow \quad x=6.$$

4 Solutions to the Exercises

Exercise 1.1 Evaluating an expression

1. $\dfrac{x+y}{2}$ for $x = 3$, $y = 7$.
   Solution: $\frac{3+7}{2} = \frac{10}{2} = 5$.

2. $5x^2 + 2y$ for $x = -2$, $y = 4$. **Solution:** $5(-2)^2 + 2(4) = 5 \cdot 4 + 8 = 28$.

3. $\dfrac{x^2 - y^2}{x - y}$ for $x = 5$, $y = 2$. **Solution:** $\frac{25-4}{5-2} = \frac{21}{3} = 7$.

Exercise 2.1: Collecting like terms

1. $6a^2 + 3a^2 = 9a^2$.
2. $4x - x + 3x = (4 - 1 + 3)x = 6x$.
3. $10 - 3 = 7$.
4. $5y + 3x - 2y = (5y - 2y) + 3x = 3y + 3x$.

Exercise 2.2: Expanding and simplifying

1. $3(x + 2) + 7 = 3x + 6 + 7 = 3x + 13$.
2. $4(3y - 1) - 2y = 12y - 4 - 2y = 10y - 4$.
3. $-5(2a + 3) - 4a = -10a - 15 - 4a = -14a - 15$.
4. $2(m - 4) + 3(m + 5) = 2m - 8 + 3m + 15 = 5m + 7$.

Exercise 3.1: Solutions of an equation

Determine whether or not $x = -2$ and $x = 2$ are solutions of $x^2 - 5 = 4x + 7$.

- If $x = -2$: $x^2 - 5 = 4 - 5 = -1$ and $4x + 7 = -8 + 7 = -1$. The equation is true, so $x = -2$ **is** a solution.
- If $x = 2$: $x^2 - 5 = 4 - 5 = -1$ and $4x + 7 = 8 + 7 = 15$. The equation is false, so $x = 2$ **is not** a solution.