MAT140 — Lecture 3 Handout

An Introduction to Linear Equations

Last time we introduced:

- **Expressions** (like "words/phrases"), e.g. $4x + 3$, and
- **Equations** (like "sentences"), e.g. $4x + 3 = 0$.

We also saw how to simplify expressions by using properties of real numbers (distributivity, commutativity, associativity) and by collecting like terms.

We are now ready to use these techniques to actually **solve** equations. Throughout this and the next lecture, we will focus on *linear equations* in one variable. As you will see, these types of equations are just about as simple as we can make an equation. Generally, this is a good approach for learning mathematics: start simple, and slowly increase the complexity as we go. That way, any complex problems can be solved by reducing them to the simple problems we know how to do. You will see this pattern repeatedly throughout these notes.

1 Solving linear equations

1.1 What does it mean to "solve" an equation?

Before getting to solutions of equations, we should first recall what "solving" an equation actually means.

> Solving an equation
>
> To solve an equation means to find the value(s) of the variable that make the equation **true**.

Generally there are two important stages of solving equations:

- Stage 1: **Find** the solution(s).
- Stage 2: **Check** each solution.

Double-checking your proposed solutions is easy and helpful: you never know if you have accidentally made a human error, so checking your solution is an easy way to avoid making mistakes.

1.2 Linear equations in standard form

A **linear equation** in one variable is an equation where the variable appears only to the first power (and is not multiplied by itself, inside an exponent, etc.). As we will see later in the course, there is a very good reason for this name. For now, however, we will merely focus on defining the simplest version of a linear equation.

Linear equation (in one variable)

A **linear equation in standard form** is an equation of the form $ax + b = 0$, where a and b are real numbers and a is non-zero.

In order to solve a linear equation in standard form, we need to rearrange the equation to isolate x (i.e. to get x on its own). In order to do that, we follow the rule:

The balancing rule

"**Whatever you do to the left, you must do to the right.**"

As an example, consider the equation $3x - 15 = 0$. We can isolate x by identifying the order of operations: the expression $3x - 15$ is saying that we first multiply x by 3, and then we subtract 15 from it. So, in order to get x on its own, we need to do the opposite of this:

$$\begin{aligned} 3x - 15 &= 0 \\ 3x &= 15 && \text{(add 15 to both sides)} \\ x &= 5 && \text{(divide both sides by 3)} \end{aligned}$$

We may then double-check that 5 is indeed the solution by substituting $x = 5$ into the original equation: $3(5) - 15 = 15 - 15 = 0$.

A General Solution

In fact, linear equations in standard form have a general solution:

$$ax + b = 0 \implies ax = -b \implies x = -\frac{b}{a}.$$

Exercise 1.1

Solve each equation.

(a) $2x + 18 = 0$
(b) $6x - 15 = 0$
(c) $-2x + 9 = 0$
(d) $\frac{x}{3} + 7 = 0$
(e) $0.4x + 2.4 = 0$

2 Linear Equations in Non-Standard Form

Sometimes a linear equation does not look exactly like $ax + b = 0$. In that case, we start by simplifying the expressions down a bit (by using distributivity, combining like terms, etc.), and only then do we isolate the variable.

Example 2:

Consider the equation $2(x-2)+5x=3x+16$. Here, we have many copies of x, so we simplify things first by expanding out the brackets and collecting like terms together. After that, things become a lot clearer:

$$
\begin{aligned}
2(x-2)+5x &= 3x+16 && \text{Write the original equation} \\
2x-4+5x &= 3x+16 && \text{Expand the brackets} \\
7x-4 &= 3x+16 && \text{Collect like terms} \\
4x-4 &= 16 && \text{Subtract } 3x \text{ from both sides} \\
4x &= 20 && \text{Add 4 to both sides} \\
x &= 5 && \text{Divide both sides by 4}
\end{aligned}
$$

Exercise 2.1

Solve $2x+3=2(x+4)$.

Exercise 2.2

Solve $4(x+3)=4x+12$.

> Uniqueness of Solutions
>
> If a linear equation is in standard form, then there will be a unique solution. However, if a linear equation is *not* in standard form, then there may be either a unique solution, infinitely-many solutions, or even no solution at all.

3 Word Problems

Throughout the next two lectures we will explore several real-world applications of linear equations. All of these problems have the same conceptual flow:

1. Extract the key information from the wording of the question.
   - What is the question asking you to calculate?
   - What information does the question give you?
2. Figure out the relationship between the starting information and the desired quantity you are being asked to calculate.
3. Write this relationship in the form of an algebraic equation, where the unknown quantity is represented by a variable.
4. Solve the algebraic equation to find the exact value of the unknown quantity.
5. Communicate the results using English.

Here is an example of this workflow.

Example 3.1

You have 96 meters of fencing to enclose a rectangular pen for your dog. The rectangle should be three times as long as it is wide. Find the dimensions of the rectangular pen.

1. The question is asking us to find the width and length of the rectangle built from 96 m of material. We are only told that the length of the rectangle should be three times the width.
2. The perimeter of a polygon is always the sum of the sides. Since we are working with a rectangle, we know that the perimeter will be twice the length plus twice the width.
3. We know that the perimeter is 96. Let the width of the rectangle be denoted by x, which is currently unknown. We know that the length is therefore $3x$, because it's assumed to be three times the width. The equation relating the perimeter to the width and length is: $96 = 2(x) + 2(3x)$.
4. We can solve the equation by simplifying:

$$\begin{aligned}96 &= 2(x) + 2(3x)\\96 &= 2x + 6x\\96 &= 8x\\12 &= x\end{aligned}$$

5. Therefore, in order to create a rectangular pen whose length is three times its width with 96 m of total fencing, we would need to make the width equal to 12 m.

Exercise 3.2

A job provides 24 paychecks a year plus an annual bonus of $750. If the yearly total is $40,830, how much is each paycheck?

4 Equations that Reduce to Linear Form

4.1 Grouping symbols (parentheses)

Guidelines (grouping)

To solve a linear equation with grouping:

1. Simplify (distribute, combine like terms).
2. Isolate the variable using properties of equality.
3. Check your solution.

As an example of this, consider the equation $4(x - 3) = 8$. Here we have an equation that certainly *appears* to be linear. We can solve it by applying distributivity to the left hand side, and then rearranging as we have seen previously:

$$
\begin{aligned}
4(x-3) &= 8 && \text{Write original equation}\\
4\cdot x - 4\cdot 3 &= 8 && \text{Apply the distributive property}\\
4x - 12 &= 8 && \text{Simplify}\\
4x - 12 + 12 &= 8 + 12 && \text{Add 12 to each side}\\
4x &= 20 && \text{Combine like terms}\\
\frac{4x}{4} &= \frac{20}{4} && \text{Divide each side by 4}\\
x &= 5 && \text{Simplify}
\end{aligned}
$$

Of course, we ought to double-check our answer: $4(5-3) = 4(2) = 8$.

4.2 Equations involving fractions

We can also make linear equations in non-standard form that involve fractions. In order to solve this type of equation, we need to clear the denominators by multiplying the equation by a nice-enough number. Typically, this will be the least common multiple (LCM) of the denominators on either side of the equation.

For example, consider the equation $\frac{x+3}{4} = \frac{x-1}{2}$. Since the least common multiple of 2 and 4 is 4, we can solve by multiplying both sides of the equation by 4:

$$
\begin{aligned}
\frac{x+3}{4} &= \frac{x-1}{2}\\
4\cdot\frac{x+3}{4} &= 4\cdot\frac{x-1}{2}\\
x+3 &= 2(x-1)\\
x+3 &= 2x-2\\
x &= 5
\end{aligned}
$$

Cross-multiplication (same idea)

When an equation equates two fractions, you can treat them as equivalent fractions and use **cross-multiplication**:

$$\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc.$$

This is just another way to clear denominators while keeping the equation balanced.

Exercise 4.1

Solve $\frac{x}{5} + \frac{3x}{4} = 19$.

4.3 Equations involving decimals

Decimals are just an expansion of several fractions involving powers of 10. For example:

$$2.13 = \frac{213}{100} = 2 + \frac{1}{10} + \frac{3}{100}.$$

If we have a linear equation involving decimals, then we can remove the decimals by multiplying by a sufficiently high power of 10. For example, consider the equation $0.3x + 0.2(10 - x) = 0.15(30)$.

The decimals can be removed by multiplying both sides by 100:

$0.3x + 0.2(10 - x) = 0.15(30)$	Write original equation
$100\big(0.3x + 0.2(10 - x)\big) = 100\big(0.15(30)\big)$	Multiply each side by 100
$30x + 20(10 - x) = 15(30)$	Remove decimals
$30x + 200 - 20x = 450$	Use the distributive property
$10x + 200 = 450$	Combine like terms
$10x = 250$	Subtract 200 from each side
$x = 25$	Divide each side by 10

Exercise 5.1

Solve $0.9x - 0.5 = 0.4x + 0.7$

5 Percentages and discount problems

The word "cent" in Latin means "one hundred" – this is where we get words like *century* or *centimeter*. Similarly, the word "percent" just means "per hundred". Any percentage $p\%$ can be readily expressed as a fraction or decimal form as follows.

5.1 Converting percentages

Percent means "per hundred"

$$p\% = \frac{p}{100}$$

For example:

$$25\% = \frac{25}{100} = 0.25.$$

Importantly, 100% has the decimal form 1. We can use this and other common forms:

- $50\% = \frac{1}{2} = 0.5$
- $25\% = \frac{1}{4} = 0.25$
- $10\% = \frac{1}{10} = 0.1$
- $1\% = \frac{1}{100} = 0.01$

to estimate other percentages before calculating. We can also convert decimals to percentages by multiplying by 100, which moves the decimal point two places to the right, for example:

$$0.33 \times 100 = 33\%.$$

For fractions, we need to first convert the fraction into one with the denominator equal to 100. The numerator will then be the percentage.

Exercise 6.1

(a) Convert 3.5% to a decimal.
(b) Convert 55% to a fraction in simplest form.

5.2 The basic percent equation

Percent equation

If "a is $p\%$ of b," then

$$a = \left(\frac{p}{100}\right) b.$$

The percentage equation is a linear equation that can be easily evaluated.

Exercise 6.2

What number is 30% of 70?

Exercise 6.3

A person makes $40,000 per year and gets a 5% raise. What is the new salary?

Exercise 6.4

14 is 25% of what number?

5.3 A neat trick

A neat trick: symmetry of percentages

If "a is $p\%$ of b," then

$$a = \left(\frac{p}{100}\right) b = \frac{1}{100}(pb) = \frac{b}{100}p,$$

so a is also $b\%$ of p.

For example: 7% of 100 is 100% of 7.

Exercise 6.5

Compute 63% of 90.

Solutions to the Exercises

Exercise 1.1 (standard form)

(a) $2x + 18 = 0 \Rightarrow 2x = -18 \Rightarrow x = -9.$
(b) $6x - 15 = 0 \Rightarrow 6x = 15 \Rightarrow x = \frac{15}{6} = \frac{5}{2}.$
(c) $-2x + 9 = 0 \Rightarrow -2x = -9 \Rightarrow x = \frac{9}{2}.$
(d) $\frac{x}{3} + 7 = 0 \Rightarrow \frac{x}{3} = -7 \Rightarrow x = -21.$
(e) $0.4x + 2.4 = 0 \Rightarrow 0.4x = -2.4 \Rightarrow x = \frac{-2.4}{0.4} = -6.$

Exercise 2.1 (a contradiction)

$$\begin{aligned} 2x + 3 &= 2(x + 4) \\ 2x + 3 &= 2x + 8 \\ 3 &= 8 \end{aligned}$$

This is false, so there is **no solution**.

Exercise 2.2 (infinitely many solutions)

$$4(x + 3) = 4x + 12 \;\Rightarrow\; 4x + 12 = 4x + 12,$$

which is always true, so there are **infinitely many solutions**.

Example 3.1 (geometry)

Let width $= w$, length $= L$.

$$2L + 2w = 96, \qquad L = 3w.$$

Substitute $L = 3w$:

$$2(3w) + 2w = 96 \Rightarrow 6w + 2w = 96 \Rightarrow 8w = 96 \Rightarrow w = 12.$$

Then $L = 3w = 36$.

$$\boxed{\text{Width } = 12 \text{ m, Length } = 36 \text{ m.}}$$

Exercise 3.2 (salary + bonus)

Let x be each paycheck:

$$24x + 750 = 40830 \Rightarrow 24x = 40080 \Rightarrow x = \frac{40080}{24} = 1670.$$

$$\boxed{\text{Each paycheck is \$1670.}}$$

Exercise 4.1

LCD of 5 and 4 is 20. Multiply both sides by 20:

$$20\left(\frac{x}{5}+\frac{3x}{4}\right)=20(19)\Rightarrow 4x+15x=380\Rightarrow 19x=380\Rightarrow x=20.$$

Exercise 5.1

Multiply both sides by 10:

$$\begin{aligned}0.9x-0.5&=0.4x+0.7\\9x-5&=4x+7\\5x-5&=7\\5x&=12\\x&=2.4\end{aligned}$$

Exercise 6.1

(a) $3.5\%=\frac{3.5}{100}=0.035.$

(b) $55\%=\frac{55}{100}=\frac{11}{20}.$

Exercise 6.2

$$x=0.30\cdot 70=21.$$

Exercise 6.3

Raise $=0.05\cdot 40000=2000$. New salary $=40000+2000=42000$.

Exercise 6.4

$$14=0.25x\Rightarrow x=\frac{14}{0.25}=56.$$

Exercise 6.5

$$63\%\text{ of }90=90\%\text{ of }63=0.90\cdot 63=56.7.$$