MAT140 — Lecture 4 Handout

More on Linear Equations

Last lecture we introduced *linear equations*, which are pretty much the simplest possible types of equations. We started with their basic form: $ax + b = 0$ and then progressively made them more complicated. Throughout that lecture, we learned how to solve all these different types of linear equations.

In this lecture we will start to look at some applications of linear equations. Specifically, we will develop several standard tools that come up constantly when working with linear relationships. We will see:

1. ratios, unit rates, and proportions,
2. some basic real-world applications of linear relationships,
3. linear inequalities and interval notation, and finally
4. absolute value and absolute value inequalities.

1 Ratios and Proportions

1.1 Ratios

The word "ratio" comes from Latin, and means something like "calculation" or "proportion". In fact, we have seen this root word being used already: the term "rational number" refers to real numbers that can be represented as a *ratio* of two numbers, for instance fractions. This should give a rather large clue for the following definition.

Ratio

A **ratio** is a comparison of two quantities using division. If a and b are numbers with $b \neq 0$, then the ratio of a to b can be written as

$$\frac{a}{b} \qquad \text{or} \qquad a : b.$$

The ratio notation $a : b$ is the same as the fraction $\frac{a}{b}$. Therefore, the order is important – writing $a : b$ is generally *not* the same as writing $b : a$.

When solving problems related to ratios, often the easiest thing to do is to convert everything to fractions and work from there. Therefore, this is a basic skill when working with ratios. Here are some examples:

1. The ratio of 7 to 5 is given by $\frac{7}{5}$.
2. The ratio of 12 to 8 is given by $\frac{12}{8} = \frac{3}{2}$
3. The ratio of 10 to 2 is given by $\frac{10}{2} = \frac{5}{1}$
4. The ratio of $3\frac{1}{2}$ to $5\frac{1}{4}$ is given by $\frac{7}{2} \div \frac{21}{4} = \frac{7}{2} \times \frac{4}{21} = \frac{28}{42} = \frac{2}{3}$

In practice, ratios typically come with units attached. For example, we could be taking a ratio to compare lengths, or weights, or times, or speeds, etc. In real-life situations like this, it is important to make sure that the quantities you are comparing are in the same units. For example: if you are trying to compare 4 feet to 8 inches, then you would need to first convert these two quantities into the same unit of measurement, e.g. by turning the 4 feet into 48 inches. Only then does the ratio become meaningful.

Exercise 1

Find ratios to compare the relative sizes of the following:

1. 5 litres to 7 litres
2. 3 meters to 40 centimeters
3. 200 cents to 3 dollars
4. 30 months to 1.5 years

1.2 Unit Rates and Unit Price

For our first application, we will now use ratios to determine the unit prices of goods being sold. The general formula is given as follows.

Unit rate (unit price)

A **unit rate** is a rate per *one* unit. For a price comparison, the **unit price** is

$$\text{unit price} = \frac{\text{cost}}{\text{quantity}}.$$

As an example, suppose that a shop is selling two bags of rice, which we will call *Bag A* and *Bag B*. Suppose that Bag A weighs 5kg, and costs 4500 yen, whereas Bag B weighs 7kg and costs 6000 yen. Which is the better deal?

We can compare the two bags by creating a ratio of their costs per weight. There are several ways to do this – we may either use kilograms or grams, so to illustrate the point we will do both. If we wanted to create a ratio of "cost per kilogram" then we would use our unit price formula as follows:

$$\text{Bag A: } \frac{4500}{5} = 900 \text{ yen/kg}, \qquad \text{Bag B: } \frac{6000}{7} \approx 857.14 \text{ yen/kg}.$$

Alternatively, we could also compare these brands by calculating the "cost per gram":

$$\text{Bag A: } \frac{4500}{5000} = \frac{9}{10} \text{ yen/g}, \qquad \text{Bag B: } \frac{6000}{7000} = \frac{6}{7} \text{ yen/g}.$$

In either case, we see that Bag B has a lower cost per weight, and therefore it is the better deal.

Exercise 2

Which has the lower unit price: a 12-ounce box of breakfast cereal for \$2.79 or a 16-ounce box of the same cereal for \$3.49?

1.3 Proportions

A proportion is simply a statement that equates two ratios, i.e. an equation consisting of two ratios on either side.

Proportion

A **proportion** is an equation stating that two ratios are equal, i.e.

$$\frac{a}{b} = \frac{c}{d} \quad \text{where } b \neq 0 \text{ and} d \neq 0.$$

In practice, problems involving proportions will often involve an unknown quantity that needs to be determined. The easiest method to do this is via cross-multiplication.

Cross-multiplication

If $\frac{a}{b} = \frac{c}{d}$ (with $b \neq 0$ and $d \neq 0$), then

$$ad = bc.$$

The method of cross-multiplication is simply a multiplication by both denominators at the same time:

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} && \text{Multiply by } bd \\ bd\left(\frac{a}{b}\right) &= bd\left(\frac{c}{d}\right) && \text{Simplify} \\ ad &= bc \end{aligned}$$

As an example, consider the proportion:

$$\frac{50}{x} = \frac{2}{28}.$$

We can solve for x by cross-multiplying:

$$\begin{aligned} \frac{50}{x} &= \frac{2}{28} \\ 50\,(28) &= 2x \\ \frac{1400}{2} &= x \\ 700 &= x \end{aligned}$$

Exercise 3

Solve $\frac{x-2}{5} = \frac{4}{3}$.

1.3.1 Scale Drawings

Proportions are useful when making scale drawings of shapes. Consider the following example: a triangular lot has perpendicular sides of 100 ft and 210 ft. Suppose that you make a proportional sketch using 8 inches for the shorter side. If you want your drawing to be proportional to the larger lot, then how long should the longer side be?

We can solve this problem by setting up and solving a proportion. Firstly, we note that the large triangular lot has a ratio of side lengths $210 : 100$, and the unknown quantity x makes our drawing have a ratio of side lengths $x : 8$. Our question is asking us to ensure that these two ratios are the same. So, we write out a proportion and solve for x:

$$\begin{aligned} \frac{210}{100} &= \frac{x}{8} \\ 8 \times 210 &= 100x \\ \frac{1680}{100} &= x \\ 16.8 &= x. \end{aligned}$$

Therefore, in order to guarantee that our drawing is accurate, we need to make the longer side of the sketch 16.8 inches.

2 Word Problems and Common Formulas

Common Linear Equations formulas

Many common real-world formulas are actually linear equations.

- **Temperature:** $F = \frac{9}{5}C + 32$
- **Simple interest:** Interest = Principal Investment × Interest Rate × time
- **Distance:** distance = speed × time

We will now review several applications of these formulas.

2.1 Example – Calculating simple interest

Suppose that you deposit \$5000 into an account paying simple interest. After 6 months, the account has earned \$162.50. What is the annual interest rate?

In order to calculate this, we can use the formula above, in symbols: $I = Prt$. The information in the question provides us with

- the interest earned: $I = 162.50$,
- the principal investment: $P = 5000$, and
- the time taken: $t = 6$ months.

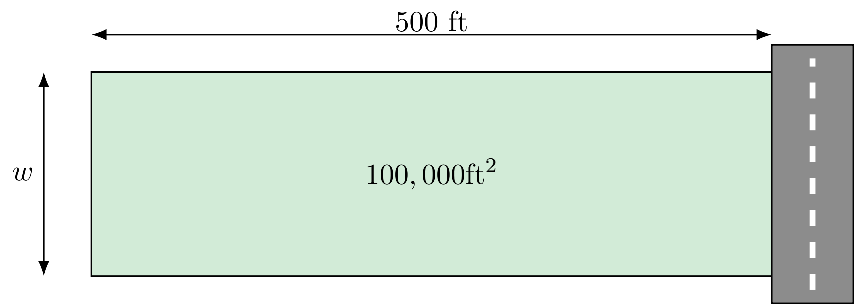
But, we are trying to calculate the *annual* interest rate, which means that the value t is actually 0.5, since 6 months is half of a year. We can now substitute the values of I, P and t into the formula and solve for r:

$$
\begin{aligned}
I &= Prt \\
162.50 &= 5000(r)(0.5) \\
162.50 &= 2500r \\
r &= \frac{162.50}{2500} = 0.065
\end{aligned}
$$

So the annual interest rate is $r = 0.065$, which is 6.5%.

2.2 Example - The fence and the field

You own a rectangular field that is 500 ft long and has a total area of 100,000 square feet. You would like to put up a fence along the frontal edge of the field, labelled w in the picture below. Fencing costs \$5.50 per foot. Based on this information, how much will the frontal length of the rectangular lot cost?



In order to solve this question we need to (a) find the value of the frontal length, and then (b) calculate its cost. We will do this separately:

(a) Find the frontal length. Using $A = lw$:

$$
\begin{aligned}
A &= lw \\
100{,}000 &= 500(w) \\
w &= \frac{100{,}000}{500} = 200.
\end{aligned}
$$

So the frontage is 200 ft.

(b) Total assessment:

$$200(5.50) = 1100.$$

So the assessment is \$1100.

2.3 Example – The leaky pool

An above-ground swimming pool has a capacity of 15,600 gallons. A drain pipe can empty the pool in 6.5 hours. At what rate (in gallons per minute) does the water flow through the drain pipe?

To answer this question, again we will set up a linear equation. Note that here we are being asked to express the rate in gallons *per minute*, therefore we need to first convert 6.5 hours into minutes: $6.5 \times 60 = 390$ minutes. So, we set $t = 390$. The rate at which the pool drains can be calculated in a very similar manner to the speed equation listed above: total loss of volume will equal the rate of loss multiplied by the time taken. We know that the pool fully drains in 390 minutes, so we can set up a basic linear equation and solve for the rate of loss:

$$\begin{aligned} 15600 &= 390r \\ \frac{15600}{390} &= r \\ r &= 40. \end{aligned}$$

Therefore, the pool drains at a rate of 40 gallons per minute.

Dimensions of Units

When solving real-world problems, it is important to keep track of the *dimensions* of the units that you are measuring in. For example:

- **Perimeter** is measured in **linear units** (meters, inches, feet, etc. ...).
- **Area** is measured in **square units** (meters squared, square inches, square feet, etc. ...).
- **Volume** is measured in **cubic units** (meters cubed, cubic inches, cubic feet, etc. ...).

3 Linear Inequalities

3.1 Inequalities and Solution Sets

An equation states that two algebraic expressions are the same, i.e. they are equal. In contrast, an *in*equality is a statement that two algebraic expressions are *not* equal. Since all real numbers live on the number line, they can be ordered under the less-than relation $<$. An inequality generalises this observation slightly.

Inequality

An **inequality** compares two algebraic expressions using the symbols

$$<,\ \leq,\ >,\ \geq .$$

A **solution** to an inequality is a value of x that makes the inequality true.

Typically, inequalities will not have a unique solution, but will instead have a *solution set*. Inequalities are less restrictive than equations, and therefore the solution sets often contain infinitely-many solutions.

3.2 Interval notation

The solution sets of an inequality tend to be portions of the real line known as *intervals*. We will now introduce some new notation.

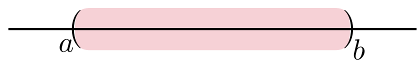
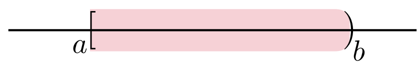
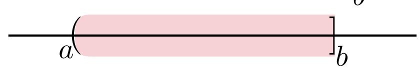
Interval notation

There are four types of bounded intervals:

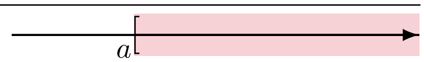
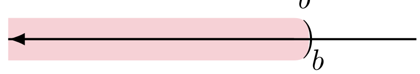
$$[a,b] = \{x : \ a \le x \le b\},$$
$$(a,b) = \{x : \ a < x < b\},$$
$$[a,b) = \{x : \ a \le x < b\},$$
$$(a,b] = \{x : \ a < x \le b\}.$$

Here, the square brackets [and] mean that an endpoint is included, and the round brackets (and) mean that an endpoint is excluded.

The bounded intervals can be represented graphically as follows.

Interval	Inequality	Number line
$[a,b]$	$a \le x \le b$	
(a,b)	$a < x < b$	
$[a,b)$	$a \le x < b$	
$(a,b]$	$a < x \le b$	

Alongside the bounded intervals detailed above, there are also *unbounded* intervals, in which we allow one of the sides of the inequality to run off to either positive or negative infinity (or both). These are pictured below.

Interval	Inequality	Number line
$[a,\infty)$	$x \ge a$	
(a,∞)	$x > a$	
$(-\infty,b]$	$x \le b$	
$(-\infty,b)$	$x < b$	
$(-\infty,\infty)$	all real numbers	

Exercise 4

Graph each inequality on a number line:

$$\text{(a) } x > -2, \qquad \text{(b) } -6 \le x \le 5, \qquad \text{(c) } x < 3.$$

3.3 Solving inequalities

Inequalities can be solved by using lots of the techniques that we have seen previously: the goal is again to use inverse operations to isolate the variable. The following properties confirm that we can perform similar operations to those of equations.

Properties of inequalities

For real numbers a, b, c:

- If $a < b$, then $a + c < b + c$ and $a - c < b - c$.
- If $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
- If $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$ (the inequality **reverses**).
- If $a < b$ and $b < c$, then $a < c$ (transitive property).

According to the above, it may be tempting to claim that solving inequalities is basically *the same* as solving an equation. However, this is not true.

Important point

Solving an inequality is very similar to solving an equation, except for two key differences:

- If you multiply or divide by a **negative** number, you must **reverse** the inequality symbol.
- The answer is usually a **solution set**, often written using **interval notation**.

As an example, we will solve the inequality $x + 6 < 9$ and graph the solution. Firstly, we remind ourselves that the goal here is to rearrange the inequality so that we isolate x, that is, we get x on its own. In this case, we can achieve this by subtracting both sides by -6:

$$\begin{aligned} x + 6 &< 9 \\ x &< 3 \end{aligned}$$

Thus any choice of x less than 3 will satisfy the inequality $x + 6 < 9$. The solution set can be described with the notation $(-\infty, 3)$, and the graph of the solution set is below.

3.4 Compound inequalities

Compound inequality

A **compound inequality** is an inequality with two parts, such as

$$a \leq \text{expression} < b.$$

We can solve compound inequalities by performing the usual operations, but this time to all three sides simultaneously.

It is easiest to understand compound inequalities by seeing them in action. Consider the inequality $-7 \leq 5x - 2 < 8$. As always, our goal here is to perform inverse operations on the expression $5x - 2$

to isolate the x. In order to do that, we must add $+2$ and then divide by 5. We make sure to do this to both inequalities at the same time:

$$\begin{aligned} &-7 \le 5x - 2 < 8 \\ &-5 \le 5x < 10 \qquad \text{(add 2 to all parts)} \\ &-1 \le x < 2 \qquad \text{(divide all parts by 5)} \end{aligned}$$

Thus any x between -1 and 2 (including -1) will satisfy our inequality $-7 \le 5x - 2 < 8$. We write the solution set as $[-1, 2)$ and we may depict it graphically as below.

Exercise 5

Solve and write the solution set in interval notation:

$$-2 \le 3x - 2 < 2.$$

4 Absolute Value and Absolute Value Inequalities

4.1 Absolute value

If we try to recall the absolute value from Lecture 1, we might remember that the absolute value is an operation that does something like "remove the minus sign from a negative number". However, technically the absolute value operation is slightly more subtle.

Absolute value

For a real number x, the **absolute value** $|x|$ is the distance from x to 0 on the number line. Equivalently,

$$|x| = \begin{cases} x, & x \ge 0, \\ -x, & x < 0. \end{cases}$$

In the two cases above, we see that this definition of the absolute value indeed has the function of removing any minus signs present in a number. Because of the two-part definition above, when solving equations involving absolute values we actually split our process into two parts. As a simple example, consider the equation $|x| = 3$. This statement is saying something like "the variable x is precisely distance 3 away from 0". This means that there are two possible solutions: $x = 3$ and $x = -3$.

In the case that we have a more complicated expression inside the absolute value, we solve the overall equation by splitting the equation up into two separate cases. We can describe these two cases by looking at the definition above. Consider the equation $|x + 2| = 4$. Here there are two cases:[1]

[1] Compare these two cases to the definition of the absolute value above. We can write the two cases by simply replacing every "x" in the definition with the expression "$x + 2$".

- Case 1: $x + 2 = 4$
- Case 2: $-(x + 2) = 4$, which implies that $x + 2 = -4$.

These two cases need to be solved separately, which means that we will obtain two different solutions (similar to how $|x| = 3$ has $x = \pm 3$). In Case 1 we solve to see that $x = 2$, and in Case 2 we solve to see that $x = -6$. These two values of x are the two solutions to the equation $|x + 2| = 4$.[2]

This method of splitting equations involving absolute values into two separate pieces and solving separately is usually the correct approach. However, there are some exceptions to this method.

Exceptional Solutions to Absolute Value Equations

There are a few situations in which an absolute value takes on special values:

- If $|x| = 0$, then $x = 0$ is the only unique solution.
- If $|x| = a$ and $a < 0$, then the equation has no solution.

Exercise 6

Solve the following:

1. $|x - 2| = 3$
2. $|x + 4| = 3x$
3. $|3x + 8| = 0$
4. $|2x - 4| = -2$

4.2 Absolute value inequalities

Inequalities involving absolute values function similarly to the equations above. In these cases, again typically the inequality will split into two pieces that need to be solved. In this case, however, these two inequalities need to be solved at the same time via a compound inequality.

Absolute value inequalities

Let a be a real number such that $a > 0$, and let x be a variable.

$$|x| < a \iff -a < x < a, \qquad |x| \leq a \iff -a \leq x \leq a.$$

At first glance, this may appear to be a strange equality. So, we will explain it in two different ways. Firstly, let's take an algebraic approach. We know that the equation $|x| = a$ splits into two cases, namely $x = -a$ and $x = a$. Similarly, we can split the inequality $|x| < a$ into two cases:

- Case 1: $x < a$
- Case 2: $-x < a$

These two cases look exactly like the two cases of Section 4.1, except that here we have replaced the equalities with inequalities. Observe now that in Case 2, we can divide both sides by -1, which flips

[2] Of course, we can double-check by substituting $x = 2$ and $x = -6$ into the equation. We have $|(2) + 2| = |4| = 4$ and $|(-6) + 2| = |-4| = 4$.

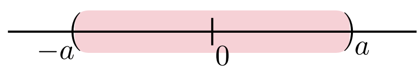
the inequality to $x > -a$. In combination with Case 1, we now have the compound inequality:

$$-a < x < a$$

We can also justify this geometrically. Recall that the absolute value gives us distance between a number on the real line and 0. In the case of the inequality $|x| < a$, we are saying something like "the distance between x and 0 is less than a". There are two options here:

- Any positive number x smaller than a will be closer to 0 than a is.
- The number $-a$ has distance $|-a| = a$ from 0, so any negative number x that is in between $-a$ and 0 will also have a distance smaller than a.

. These two are precisely the two cases above: picking an x that satisfies either $-a < x \leq 0$ or $0 \leq x < a$ will guarantee that x is closer to 0 than a or $-a$ is. Another way of writing the statement "$-a < x \leq 0$ or $0 \leq x < a$" is with the compound inequality $-a < x < a$. Graphically, any x in the shaded region will satisfy the inequality $|x| < a$:



Thus, graphically we can understand the statement $|x| < a$ as describing an interval of length $2a$ around zero. Replacing the inequality $<$ with $\leq$ will simply include the endpoints $-a$ and a in the interval.

As a matter of fact, we can "slide" these intervals of length $2a$ around by manipulating the expression $|x| < a$.

Shifted form (distance from a point)

For $a > 0$,

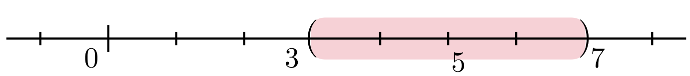
$$|x - b| < a \iff b - a < x < b + a, \qquad |x - b| \leq a \iff b - a \leq x \leq b + a.$$

This is the statement that x is within distance a of the center point b.

The number b in the inequality above allows us to recenter the interval around b instead of zero. To see this in action, let's solve the inequality $|x - 5| < 2$ and draw its solution.

$$\begin{aligned} |x - 5| &< 2 \\ -2 < x - 5 &< 2 \\ 3 < x &< 7 \end{aligned}$$

So, the solution set is the interval $(3, 7)$. Observe that the number 5 is precisely in the center of this interval, and the distance between 5 and the endpoints 3 or 7 is precisely 2. Graphically:



Solutions to the Exercises

Exercise 1

$$\text{(a) } \frac{5}{7}, \qquad \text{(b) } \frac{15}{2}, \qquad \text{(c) } \frac{2}{3}, \qquad \text{(d) } \frac{5}{3}.$$

Exercise 2

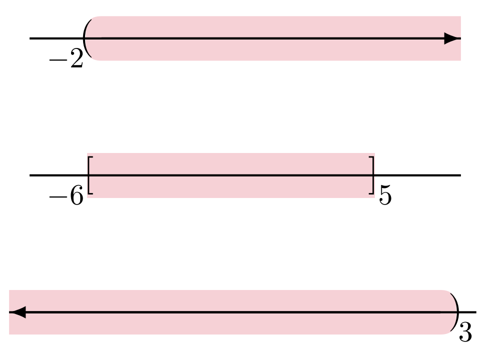
$$12 \text{ oz: } \frac{2.79}{12} \approx 0.2325 \text{ \$/oz}, \qquad 16 \text{ oz: } \frac{3.49}{16} \approx 0.2181 \text{ \$/oz}.$$

Therefore, the 16-ounce box has the lower unit price.

Exercise 3

$$\begin{aligned} \frac{x-2}{5} &= \frac{4}{3} \\ 3(x-2) &= 20 \\ 3x - 6 &= 20 \\ 3x &= 26 \\ x &= \frac{26}{3}. \end{aligned}$$

Exercise 4



Exercise 5

$$\begin{aligned} -2 \le 3x - 2 < 2 & \\ 0 \le 3x < 4 & \qquad \text{(add 2 to all parts)} \\ 0 \le x < \frac{4}{3} & \qquad \text{(divide all parts by 3)} \end{aligned}$$

Solution set: $\left[0, \frac{4}{3}\right)$.

Exercise 6

1. $|x - 2| = 3$

$$x - 2 = 3 \quad \text{or} \quad x - 2 = -3$$
$$x = 5 \quad \text{or} \quad x = -1$$

So the solutions are $x = -1, 5$.

2. $|x + 4| = 3x$
Since $|x + 4| \geq 0$, we must have $3x \geq 0$, so $x \geq 0$. Now split into cases:
Case 1: $x + 4 \geq 0$ (true for all $x \geq 0$). Then $|x + 4| = x + 4$, so

$$x + 4 = 3x \quad \Rightarrow \quad 4 = 2x \quad \Rightarrow \quad x = 2.$$

Case 2: $x + 4 < 0$, which would imply $x < -4$, contradicting $x \geq 0$. Therefore, the only solution is $x = 2$.

3. $|3x + 8| = 0$

$$3x + 8 = 0 \quad \Rightarrow \quad x = -\frac{8}{3}.$$

4. $|2x - 4| = -2$
Since $|2x - 4| \geq 0$ for all x, it cannot equal -2. Therefore, there is **no solution**.