MAT140 — Lecture 5 Handout
Graphs of Equations

Different people view mathematics in different ways. For some, mathematics is purely a "symbol game" in which the goal is to find the correct legal manipulation of symbols that will give you what you want. For others, mathematics is much more of an art – all of these symbols should have an interpretation, and it is our goal as mathematicians to discover this hidden meaning somehow. Of course, this is a pretty simplistic picture. In reality, it is useful to take *both* approaches and to be able to switch efficiently between the two. Sometimes it is helpful to crunch numbers or symbols, and other times it is helpful to try and see the bigger picture. The former mindset usually yields *results*, whereas the latter yields *insight*.

Throughout the past few lectures, we have mostly been dealing with the first mode of thinking: we have introduced a bunch of symbols and have played around with their manipulation. We saw that we can solve equations by employing a sequence of "legal moves" that leads us to a solution. In this lecture, we will start to take the opposite view by seeing what all of this *means* geometrically.

In this lecture, we learn how to interpret and create graphs of equations on the coordinate chart. We will also introduce the ideas of **relations** and **functions**, and a simple test to decide whether a graph represents a function.

1 The Coordinate Chart and Ordered Pairs

We will start with our fundamental notion for the day, that is, the *ordered pair*.

Ordered Pairs

An ordered pair is a pair of real numbers (a, b) that are ordered, meaning that

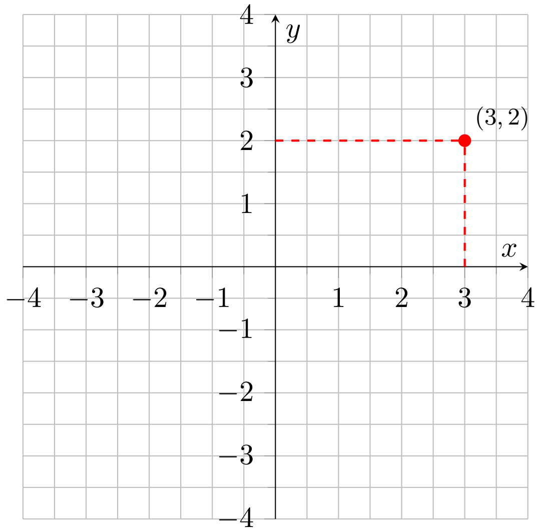
$$(a, b) \neq (b, a)$$

in general. For two ordered pairs (a, b) and (c, d), these are equal to each other only if both $a = c$ and $b = d$.

1.1 The Coordinate Chart

Real numbers can be arranged on a line, which we have been calling the *number line* or *real line*. Similarly, ordered pairs can be arranged in a *plane*, which is the Cartesian product of two copies of the real line. This allows us to interpret ordered pairs (a, b) as coordinates in this plane.

The coordinate chart, often denoted $\mathbb{R}^2$, is drawn below.



As you can see, there are two perpendicular lines, which are like two copies of the real line. These lines have labels on them, and these correspond to two fundamentally different directions: horizontal and vertical. These are called the **x-axis** and **y-axis**, respectively.

Every point in this coordinate chart can be projected down to either the x or y axes, meaning that any point in this chart can be described using two numbers: a horizontal number and a vertical number. These two numbers are called the *coordinates* of the point, and they are described with ordered pairs. For example, the ordered pair $(3, 2)$ is plotted on the graph above in red.

Given any ordered pair (x, y):

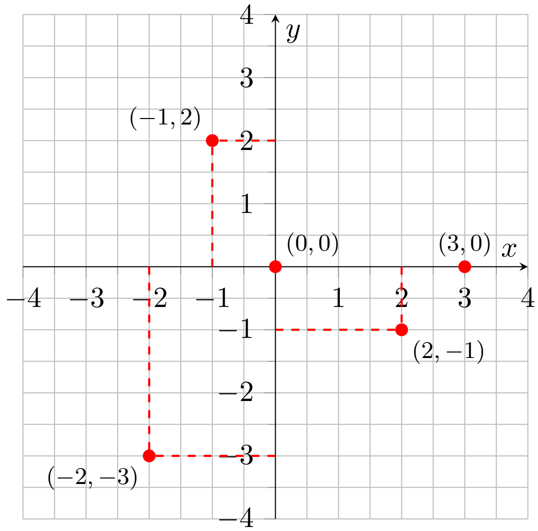
- the number x tells you how far to move left/right from the origin, and
- the number y tells you how far to move down/up from the origin.

Reading coordinates

To plot (x, y):

1. Start at $(0, 0)$.
2. Move x units along the x-axis (right if $x > 0$, left if $x < 0$).
3. Then move y units parallel to the y-axis (up if $y > 0$, down if $y < 0$).

As an example, the points $(-1, 2)$, $(3, 0)$, $(2, -1)$, $(0, 0)$, and $(-2, -3)$ are plotted on the coordinate chart below.



1.2 Ordered Pairs as Solutions to Equations

In previous lectures we have seen *solutions* to equations in one variable. Solutions to an equation are simply fixed numbers that yield a true statement once substituted into the equation. For example, the equation $x - 3 = 1$ has $x = 4$ as a solution but the value $x = 30$ is not.

When working with the coordinate chart $\mathbb{R}^2$, it is often the case that the variables x and y have a relationship that may be expressed as an equation. In contrast to those equations that we have seen in Lectures 2, 3 and 4, these equations will have *two* variables. Examples include:

$$y = 3x + 2, \qquad y + x^2 = 3, \qquad 3y + 4x = 5.$$

Solutions to Equations in Two Variables

A **solution** to an equation in two variables x and y is an ordered pair (x, y) that makes the equation true.

Let's consider the equation $y = 3x + 2$. A solution would be a choice of both an x and y value that makes the equation $y = 3x + 2$ into a true statement after substitution. For example, the ordered pair $(0, 2)$ is a solution, whereas the ordered pair $(1, 3)$ is not.[1]

[1] If we substitute $x = 0$ and $y = 2$ into the equation $y = 3x + 2$ we get: $2 = 3(0) + 2$ which is true so $(0, 2)$ is therefore a solution. However, if we substitute $x = 1$ and $y = 3$ into the equation then we get $3 = 3(1) + 2$ which is not true, so $(1, 3)$ is not a solution to the equation $y = 3x + 2$.

How to Verify a Solution

To verify that an ordered pair (a, b) is a solution of an equation with variables x and y, use these steps.

1. Substitute the values $x = a$ and $y = b$ into the equation.
2. Simplify each side of the equation.
3. If each side simplifies to the same number, then the ordered pair is a solution. If not, then the ordered pair is not a solution.

Exercise 2.1

Check whether each ordered pair is a solution to $x + 3y = 6$:

$$(1, 2), \qquad (0, 2).$$

2 Graphs of Equations in Two Variables

So far we have only seen how to verify whether an ordered pair is a solution to an equation or not. However, in general an equation with two variables will have many possible solutions. The full relationship between the two variables x and y can then be described using a graph.

The Graph of an Equation

The graph of an equation is the set of **all** solutions to the equation.

Since solutions to an equation in two variables will be ordered pairs, these graphs can be plotted on the coordinate chart $\mathbb{R}^2$. This gives an image which details the full relationship between the variables x and y. Depending on the equation, the graph may be a line, a curve, or another shape in the coordinate chart. A graph is a way to represent *all* of the solutions to an equation, all at once. Any solution to an equation will lie on its graph.

Plotting the graph of an equation may be difficult. Throughout this course we will develop various skills that will allow us to identify key components of equations and their graphs, so that we can better understand how to plot graphs accurately. For now, however, we will focus on a simpler technique known as *sketching* a graph. A sketch of a graph is like an educated guess for what the graph looks like. In fact, in some contexts it is simply good enough to provide a sketch of a graph instead of a perfectly accurate representation.

Sketching a Graph Using Points

A graph can be sketched by plotting some sample points on the coordinate chart.

1. If possible, rewrite the equation by isolating y on one side of the equality.
2. Take a sample of small, reasonable looking x values.
3. Create a table of these points, and substitute them into the equation to find the corresponding y values. These give ordered pairs which are solutions to the equation.
4. Plot these solutions on the coordinate chart $\mathbb{R}^2$.
5. Connect the points with a smooth line.

Generally, you are free to choose the number of sampling points x. However, the more values of x that you choose, the easier it will be to recognise the pattern of the graph. As a matter of fact, lots of graphing calculators will do exactly this to sketch out graphs. The computer will sample such a large number of x values that the resulting solutions *appear* as a line on the computer screen.

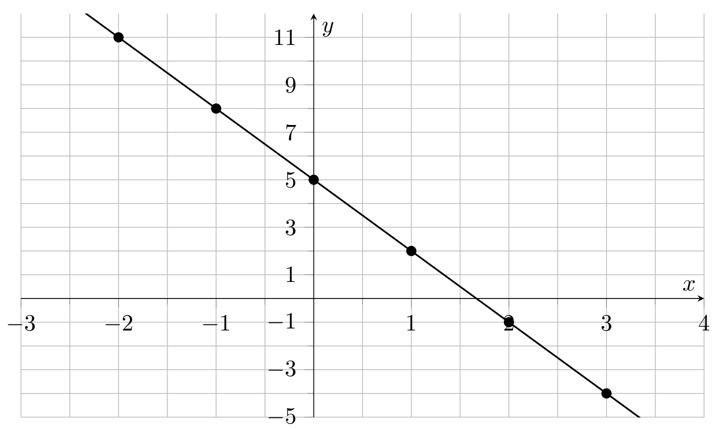
As an example of the point plot method, consider the equation $y + 3x = 5$. Here, our first step is to isolate the y variable by subtracting $3x$ from both sides. This gives us the (equivalent) equation

$$y = -3x + 5.$$

From here, we will take a sample of small values of x and fill out a table:

x	-2	-1	0	1	2	3
y	11	8	5	2	-1	-4
Solutions	$(-2, 11)$	$(-1, 8)$	$(0, 5)$	$(1, 2)$	$(2, -1)$	$(3, -4)$

We now plot the solutions in the bottom row of the table, and join them up in a line.



Exercise 2.2

Use the point plot method to sketch the graphs of:

1. $x^2 + y = 4$
2. $y = |x - 1|$

2.1 Axes Intercepts

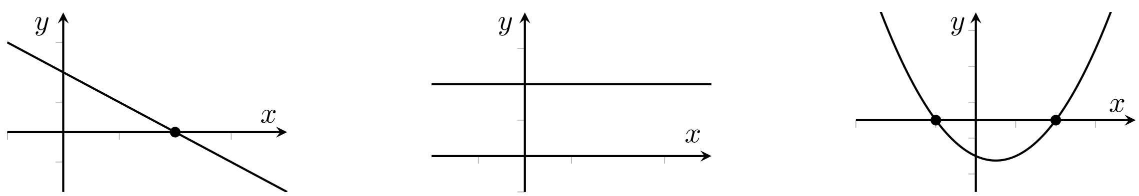
In practice, it is quite insightful to study the points where a graph crosses the x and y axes. These points are known as *intercepts*.

x and y intercepts

For the graph of any equation:

- the x **intercept** is where the graph crosses the x axis (set $y = 0$), and
- the y **intercept** is where the graph crosses the y axis (set $x = 0$).

It should be noted that depending on the shape of the graph, the intercepts with the x or y axis may or may not exist. Here are some examples of different types of x intercepts:



Unique x intercept   No x intercept   Multiple x intercepts

We can attempt to compute intercepts by setting either y or x equal to zero and solving the equation for the other variable. Depending on the equation, the required solution may not exist. As an example, consider the equation $y = 2x - 5$. Then:

$$y \text{ intercept: } x = 0 \Rightarrow y = -5 \Rightarrow (0, -5). \qquad x \text{ intercept: } y = 0 \Rightarrow 0 = 2x - 5 \Rightarrow x = \frac{5}{2} \Rightarrow \left(\frac{5}{2}, 0\right).$$

3 Relations and Functions

Relations are just a mathematical way to describe a relationship between two things. Formally, they are built using *sets*, which are actually the building blocks of many mathematical structures.

A set is an unordered, structureless collection of objects. We denote sets using curly brackets $\{$ and $\}$. This notion is so open ended that almost everything can be described using sets. For example, we could consider:

- The set of all countries on Earth: $\{\text{Afghanistan}, \text{Albania}, \ldots, \text{Zimbabwe}\}$
- The set of all moons of Jupiter: $\{\text{Europa}, \text{Io}, \text{Ganymede}, \ldots\}$
- The set of all even numbers: $\{0, 2, 4, 6, \ldots\}$.

Sets can be either finite in size or infinite, and they are only characterised by the objects that they contain. At this level, it is best to think of sets as simply “the thing that allows us to take collections of objects”.

It should be noted that we never write out members of a set more than once; for example, the set $\{1, 1, 2\}$ is the same thing as the set $\{1, 2\}$.

3.1 Relations

We may use sets to define what a relation is. There are two important components of a relation, known as the domain and the range of the relation.

Relations; Domain and Range

A relation R is a set of ordered pairs. The *domain* of R is the set of all first components of the ordered pairs in R, and the *range* of R is the set of all second components of the ordered pairs in R. We denote the domain of R by $\mathrm{Dom}(R)$, and we denote the range of R by $\mathrm{Ran}(R)$.

An example of a relation might be:

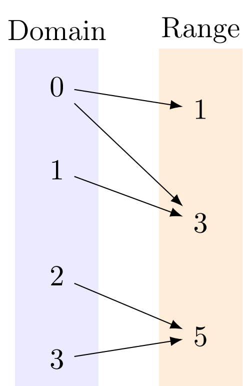
$$R = \{(0,1), (1,3), (2,5), (3,5), (0,3)\}.$$

Here, the domain of R will be the set of all first components of the pairs, and the range of R will be the set of all the second components of the pairs:

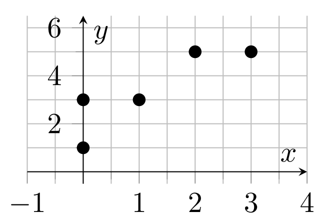
$$\mathrm{Dom}(R) = \{0,1,2,3\} \qquad \mathrm{Ran}(R) = \{1,3,5\}.$$

Observe that we do not write the same elements multiple times.

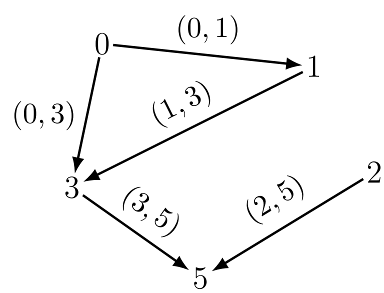
There are many ways to represent relations, and depending on the context one may be better than another. Below are three possible ways to represent a relation: as a mapping diagram, as a coordinate plot, and as an "abstract graph".



Mapping diagram



Coordinate plot



Abstract graph

3.2 Functions

A "function" is a fundamental tool in mathematics that can be used to describe all sorts of interesting things. It is best to think of a function as a sort of *machine* that takes in inputs and spits out some result. The machine has some rules: there are only certain inputs that it will accept, and to each input there is **only one output**. A more precise definition is as follows.

Function

A **function** is a relation in which no input value has two different output values.

- f is the **name** of the function,
- x is the **input** (domain value),
- $f(x)$ is the **output** (range value) corresponding to that input.

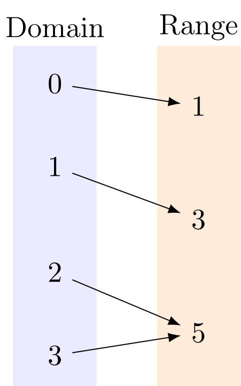
3.2.1 Identifying Functions

It is important to be able to identify functions. Depending on how a function is represented, this may amount to thinking in different ways. However, there is a fundamental rule in any case.

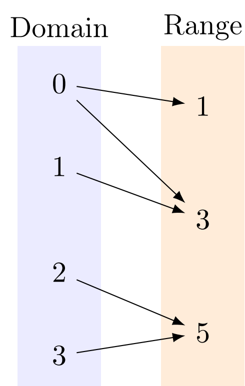
Identifying Functions

If a relation R has an element in its domain that is related to two distinct elements in its range, then R is not a function.

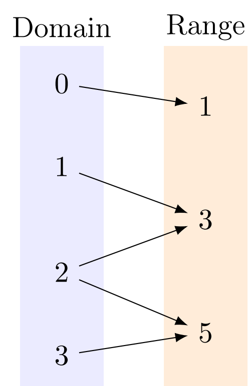
If a relation is represented as a mapping diagram, then the relation is a function if there is only one arrow coming out of each member of the domain. For example, consider the relation represented below.



Function



Not a function



Not a function

As you can see, in the second and third case there are elements of the domain that have multiple arrows coming out of them. This means that they are not functions. However, in the first case, every member of the domain is only mapped to one element of the range, so therefore this diagram represents a function.

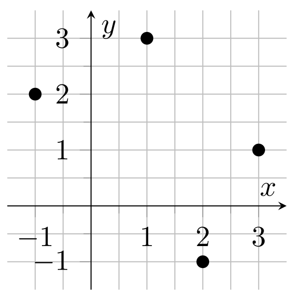
The Vertical Line Test

A relation R can also be represented as a graph in the coordinate chart by placing values from $\mathrm{Dom}(R)$ along the x axis and values from $\mathrm{Ran}(R)$ along the y axis. In this case, we can visually check if a given relation is a function by using the "vertical line test".

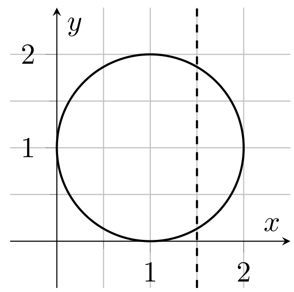
The Vertical Line Test

A graph represents y as a function of x if and only if **no vertical line** intersects the graph more than once.

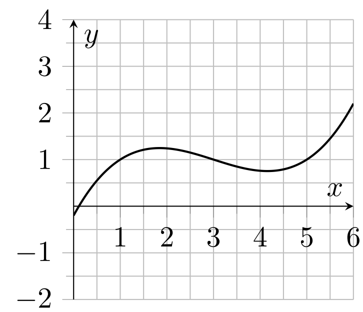
If a vertical line were to touch a graph twice, that would mean that there were two values of y that correspond to the same input x, and therefore the graph does not represent a function. Below are three examples.



Function



Not a function



Function

3.3 Evaluating a Function

The notation $f(x)$ means the output (value) of the function f when the input is x. So, we may plug in particular values of x and see what the function does to them. This is known as "evaluating" a function. As an example, consider the function $g(x) = 3x - x^2$. We evaluate g at $x = 2$ and $x = 0$:

$$g(2) = 3(2) - (2)^2 = 6 - 4 = 2, \qquad g(0) = 3(0) - (0)^2 = 0.$$

Exercise 3.1

Let $h(x) = 2x + 1$. Compute $h(-3)$ and $h(4)$.

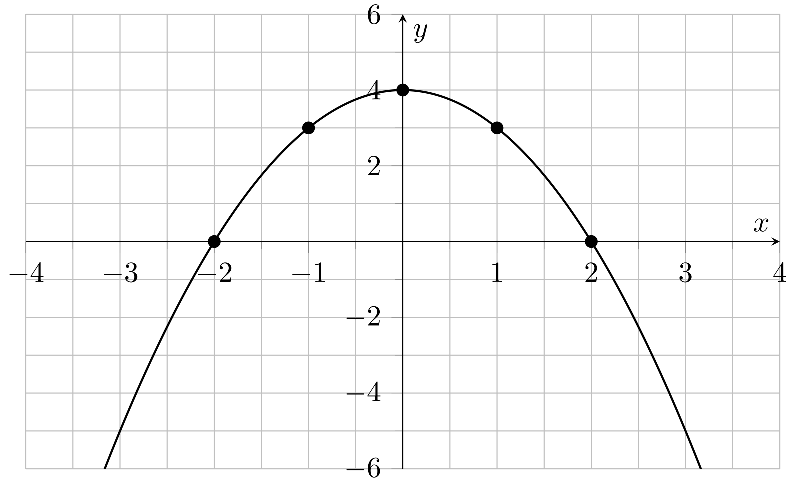
Solutions to the Exercises

Exercise 2.1

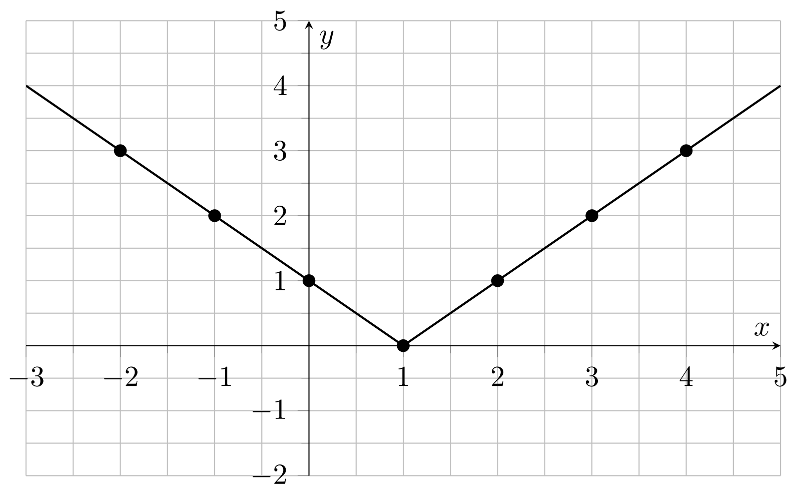
$$(1,2): \; 1 + 3(2) = 7 \neq 6 \;\Rightarrow\; \text{not a solution.} \qquad (0,2): \; 0 + 3(2) = 6 \;\Rightarrow\; \text{is a solution.}$$

Exercise 2.2

1. The equation $x^2 + y = 4$ can be rewritten as $y = 4 - x^2$, which is a downward opening parabola.



2. The absolute value function $y = |x - 1|$ makes a "V" shape with a corner at $(1, 0)$.



Exercise 3.1

$$h(-3) = 2(-3) + 1 = -6 + 1 = -5, \qquad h(4) = 2(4) + 1 = 9.$$