MAT140 — Lecture 6 Handout

Graphs of Linear Equations

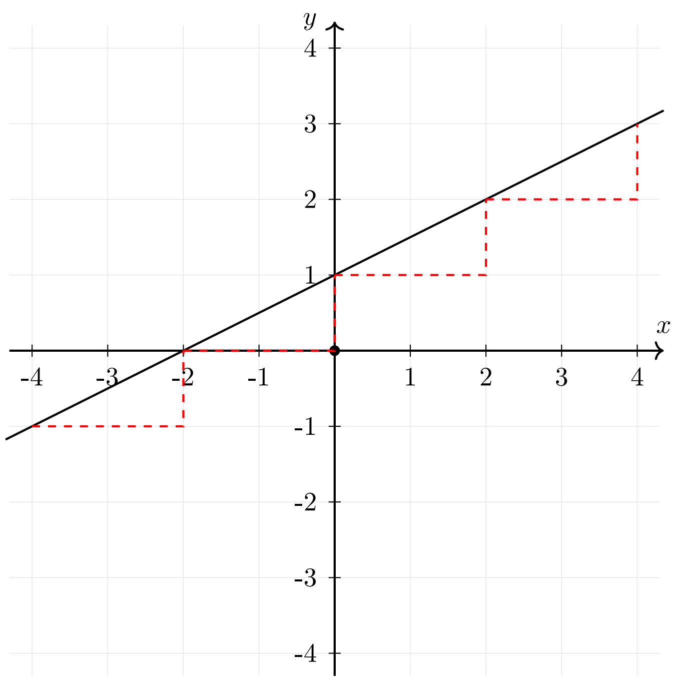
Last time we saw how to plot basic graphs from common equations by picking sample points and sketching the resulting shape. In this lecture we start to see a deeper idea: the algebraic and geometric perspectives are two sides of the same coin. This is a fundamental theme in mathematics, often called the *algebra-geometry correspondence*. Today we study the simplest case of this correspondence: *graphs of linear equations*. We will:

- introduce the notion of gradient and explain how it controls the shape of a line,
- learn how to move between equations of lines and their graphs, and
- learn how to sketch graphs of linear inequalities.

1 Gradient and Graphs of Linear Equations

1.1 What is a Gradient?

Consider the line drawn below.



Notice how the line changes at a constant rate: for every 2 squares we move to the right, the line always moves up 1 square. It doesn't matter whereabouts we start from, in any case the line always changes at the same rate. The gradient of a line captures this idea.

Gradient

The **gradient** of a non-vertical line measures its steepness.

Mathematically, the gradient is simply a number which quantifies how steep a line is. So, a line that is more steep will have a larger gradient (in some sense, at least), and a line that is less steep will have a smaller gradient.

We can describe the gradient of a straight line by comparing the change in vertical direction to the change in horizontal direction. This comparison is simply a ratio, which may be schematically written:

$$\text{Gradient} = \frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\text{Vertical change}}{\text{Horizontal change}}.$$

We can be more precise by picking two points that are on the line. Then, the gradient can be calculated using the following formula.

The Gradient Formula

If a non-vertical line passes through the points (x_1, y_1) and (x_2, y_2), then its gradient is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

In the above formula, we need to assume that the line is not vertical, since vertical lines have *no* change in their horizontal direction. This means that any pair of points on a vertical line have the same x values, i.e. $x_1 = x_2$. But, this implies that $x_2 - x_1 = 0$ and therefore we would be dividing by zero in the formula of m, which is not possible.

It is, however, completely fine for the two y-components y_1 and y_2 to be the same – this corresponds to a horizontal line in which there is *no steepness at all.*

When the formula for the gradient is used, it is important to keep track of the order of subtractions. Given two points on a line, you are free to label either of them (x_1, y_1) and the other (x_2, y_2), and it doesn't matter which order you take their difference, since:

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{-1}{-1}\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

However, once this choice of order has been made, you must make sure to form the numerator and denominator using the **same** order of subtraction.

Common Error

You may label either point as (x_1, y_1) and the other as (x_2, y_2). But once you choose an order, you must use the **same** order in the numerator and denominator.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}, \qquad \text{but} \qquad m = \frac{y_2 - y_1}{x_1 - x_2} \quad \text{and} \quad m = \frac{y_1 - y_2}{x_2 - x_1} \text{ are wrong.}$$

As an example, we will find the gradient of the line through $(-2, 0)$ and $(3, 1)$.

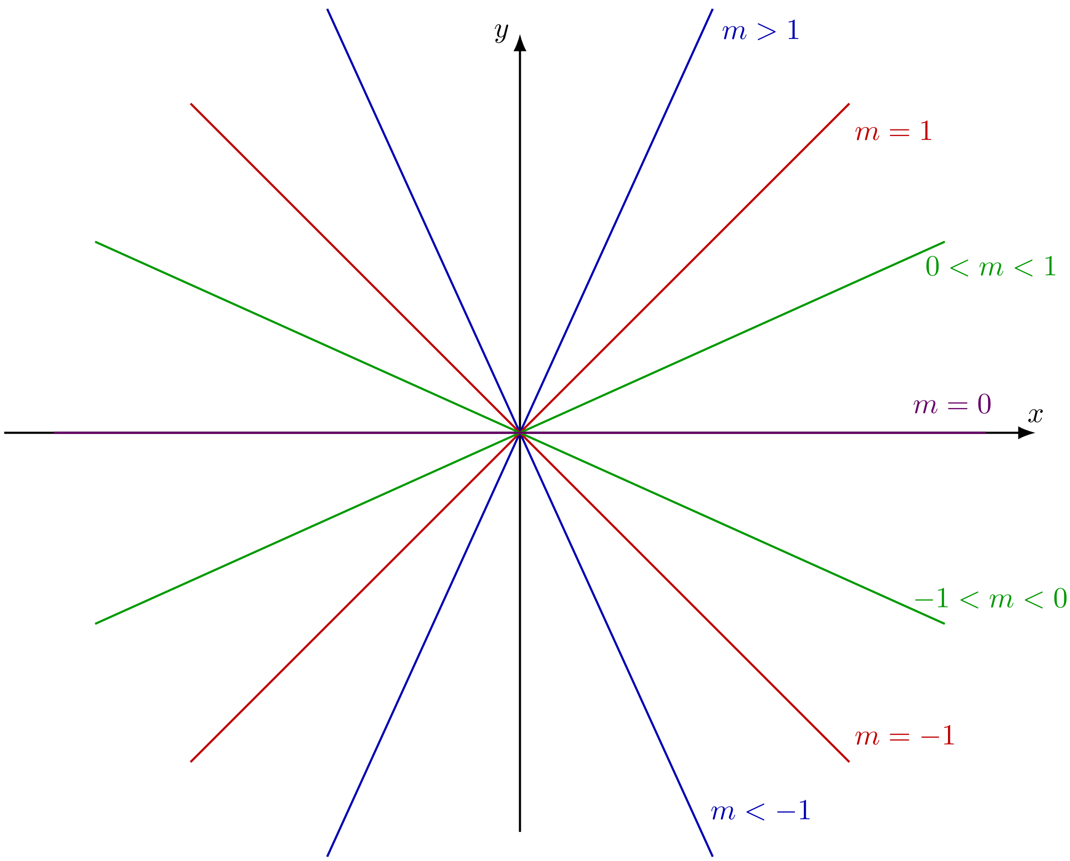
$$m = \frac{1 - 0}{3 - (-2)} = \frac{1}{5}.$$

Exercise 1

Find the gradient of the line through $(0, 0)$ and $(1, -1)$.

1.2 Gradient Visualized

The image below depicts the gradients of different lines passing through the origin. Observe that there are both positive and negative values of m, corresponding to lines that rise from left to right, and vice-versa. There is no vertical line, as that would correspond to a line with an undefined gradient.



The Sign of the Gradient

1. If $m > 0$, the line rises from left to right.
2. If $m < 0$, the line falls from left to right.
3. If $m = 0$, the line is horizontal.
4. If the line is vertical, then its gradient is **undefined**.

Exercise 2

Find the gradient of the line through each pair of points.

1. $(2, 3)$ and $(6, 7)$
2. $(-1, 4)$ and $(3, 4)$
3. $(5, -2)$ and $(5, 1)$

Exercise 3

A ladder is propped up against a tree. One end of the ladder touches the tree at a height of 12 m, and the ground 5 m away from the tree. We may consider the tree as the y-axis and the ground as the x-axis, so that the endpoints of the ladder are $(0, 12)$ and $(5, 0)$. Find the gradient of the ladder.

2 Gradients as a Graphing Aid

2.1 Gradient-Intercept form

We will now start to understand how gradients can be used to characterise straight lines. As a matter of fact, gradients *alone* will not fully characterise a line. In addition to gradients, we also need to consider where a given line crosses the y-axis.

It turns out that a non-vertical line is uniquely characterised by its gradient and y-intercept. With the Figure of Section 1.2 in mind, we can imagine the gradient as a number that tells us how much to "rotate" a horizontal line so as to change its steepness. The y-intercept then tells us how much to lift this line away from the x-axis. Mathematically, the equation for a straight line takes the gradient and y-intercept as inputs.

Gradient-intercept Form of a Line

A non-vertical line can be written as

$$y = mx + b.$$

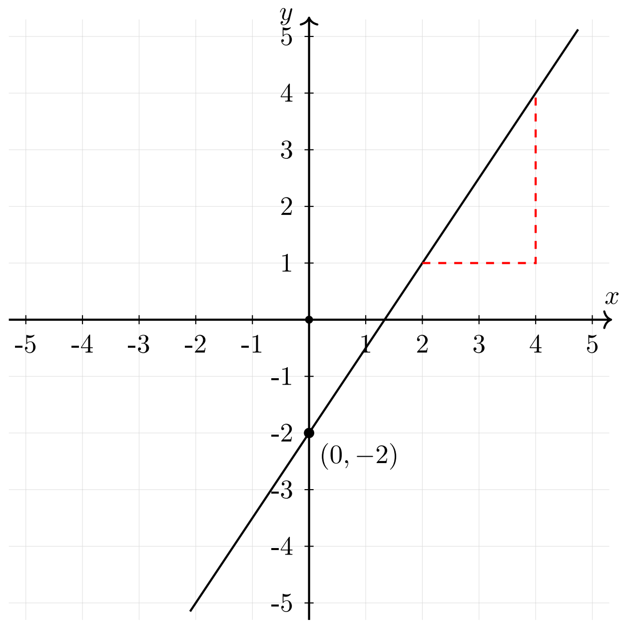
Where:

- m is the **gradient**, and
- b is the **y-intercept** (the point where the line crosses the y-axis), namely $(0, b)$.

As an example of this, consider the linear equation $3x - 2y = 4$. Here, we may rearrange to isolate y and directly read off the gradient and y-intercept of the line:

$$\begin{aligned} 3x - 2y &= 4 \\ -2y &= -3x + 4 \\ y &= \frac{3}{2}x - 2. \end{aligned}$$

So, the gradient of this line is $m = \frac{3}{2}$ and the y-intercept is $(0, -2)$. The graph of this line is below. Observe that the line moves 3 units in the vertical direction whenever there is a 2 unit change in the horizontal direction.

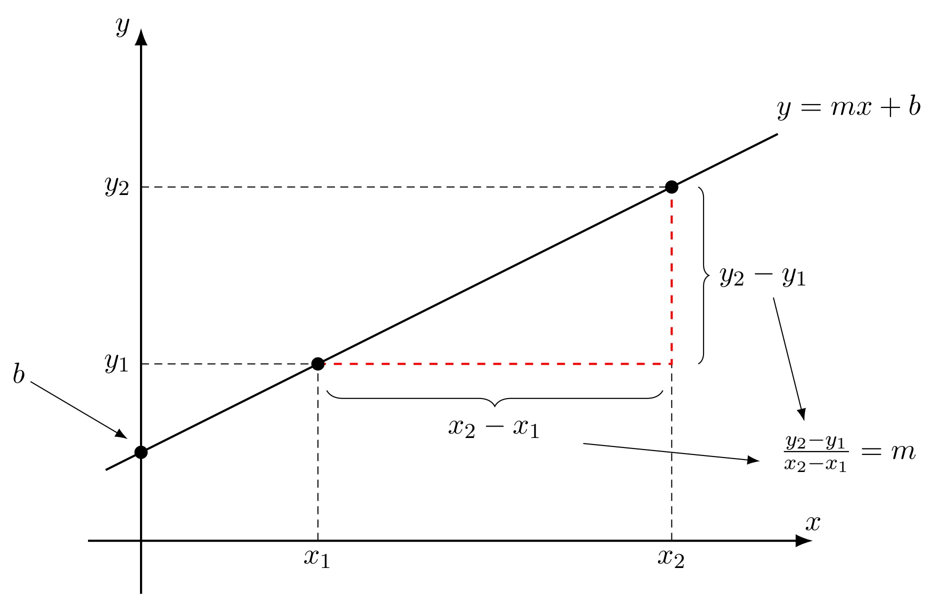


Sketching a line from an equation

To sketch the graph of $y = mx + b$:

1. Plot the y-intercept $(0, b)$.
2. Use m as a rise-over-run ratio to find a second point.
3. Draw a straight line through the two points.

The geometry of the equation $y = mx + b$ can be summarised with the following picture.



Exercise 4

Use gradient-intercept form to sketch the graph of $x - 3y = -6$.

3 Parallel and Perpendicular Lines

In the previous section we stated that the gradient alone will not uniquely describe a line. To see this, we can imagine the situation in which two lines have the same gradient, yet have different y-intercepts. In this case, the two lines would have the same steepness, but their graphs would never touch. Such an observation partly motivates the following definition.

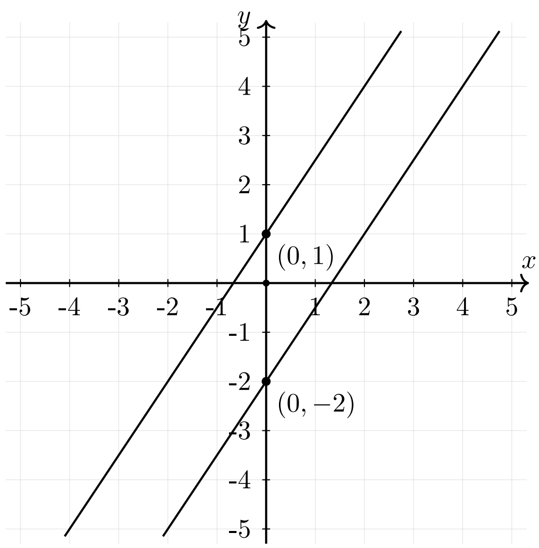
Parallel and perpendicular lines

For two non-vertical lines:

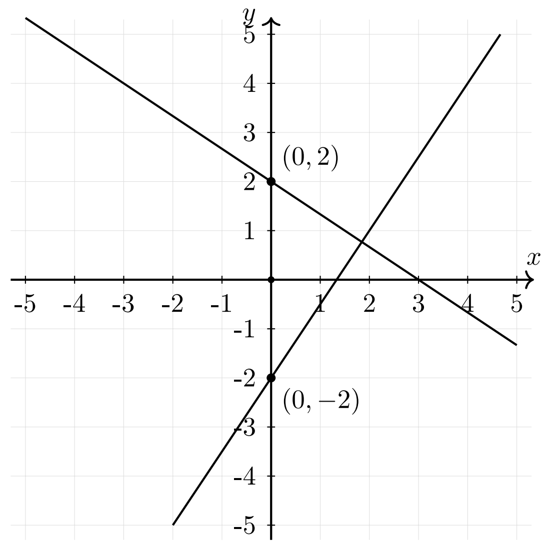
- They are **parallel** if and only if they have the **same** gradient.
- They are **perpendicular** if and only if their gradients are **negative reciprocals**.

$$m_1 = -\frac{1}{m_2}, \quad \text{where } m_2 \neq 0.$$

Geometrically, two parallel lines will never meet, whereas two perpendicular lines will intersect at right angles, as pictured below.



Parallel lines



Perpendicular lines

Exercise 5

Determine whether the lines are parallel, perpendicular, or neither:

1. the lines $y = -3x - 2$ and $y = \frac{1}{3}x + 1$.
2. the lines $y = \frac{1}{2}x + 1$ and $y = \frac{1}{2}x - 1$.

4 Equations of Lines

When discussing linear equations, there are two common types of problems:

1. given an equation, sketch its graph;
2. given a graph (or geometric data), find an equation.

We will now explore how to describe graphs of lines from partial information.

4.1 Point-Gradient form

To begin our discussion, we will now see how to describe the equation of a line with gradient m passing through a fixed point.

Point-gradient form

A line with gradient m passing through a point (x_1, y_1) has equation

$$y - y_1 = m(x - x_1).$$

Although the above formula is valid and useful, there is another way to obtain the desired equation. If we start with the equation $y = mx + b$, we recall that the graph of this equation (the line we want to describe) is precisely the collection of all points that satisfy the equation $y = mx + b$. Thus, if we are given a point (x_1, y_1), we can substitute this into the equation and solve to find the value of b.

To see an example of this strategy, we will write an equation of the line passing through the point $(1, -2)$ that has gradient $m = 3$. Using the information available to us, we know that our equation will look like $y = 3x + b$, where we need to find the value of b. So, we substitute in the point $(1, -2)$ and rearrange to find b:

$$-2 = 3(1) + b \;\Rightarrow\; b = -5.$$

Therefore, the equation for this line is $y = 3x - 5$. This can be double-checked by using the point-gradient formula listed above:

$$y - (-2) = 3(x - 1) \quad\Rightarrow\quad y + 2 = 3x - 3 \quad\Rightarrow\quad y = 3x - 5.$$

Note that this is **an** equation of the line, but not **the** equation of the line. In fact, the same line may be written in many equivalent forms. For example, the equations:

$$y = 3x - 5, \quad 3x - y = 5, \quad \text{and} \quad 3x - y - 5 = 0$$

all have the same graph.

4.2 Two-Point Form

Suppose now that we want to find the equation of a line passing through two points (x_1, y_1) and (x_2, y_2). To do this, we can first compute the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

then use the previous method. Combining these steps gives the **two-point form**:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

As an example, we will write an equation of the line passing through $(3, 1)$ and $(-3, 4)$.

First, we compute the gradient:

$$m = \frac{4 - 1}{-3 - 3} = \frac{3}{-6} = -\frac{1}{2}.$$

Now, we may pick one of the two points and solve for b:

$$1 = -\frac{1}{2}(3) + b \quad \Rightarrow \quad b = \frac{5}{2}.$$

Putting this altogether, the equation of the line passing through these two points is

$$y = -\frac{1}{2}x + \frac{5}{2}.$$

Alternatively, we may use the point-gradient form with $(x_1, y_1) = (3, 1)$:

$$y - 1 = -\frac{1}{2}(x - 3) \quad \Rightarrow \quad y - 1 = -\frac{1}{2}x + \frac{3}{2} \quad \Rightarrow \quad y = -\frac{1}{2}x + \frac{5}{2}.$$

4.3 Parallel and Perpendicular Through a Point

We can also use the point-gradient form of Section 4.1 in order to find equations of parallel or perpendicular lines passing through a given point. For example, we will derive the equations of the lines passing through the point $(2, -1)$ that are:

1. parallel to $y = \frac{2}{3}x - \frac{5}{3}$, and
2. perpendicular to $y = \frac{2}{3}x - \frac{5}{3}$.

Since parallel lines have the same gradient, we know that our first equation will have $m = \frac{2}{3}$. Either solving for b manually or using the point-gradient formula then yields:

$$y - (-1) = \frac{2}{3}(x - 2) \quad \Rightarrow \quad y + 1 = \frac{2}{3}x - \frac{4}{3} \quad \Rightarrow \quad y = \frac{2}{3}x - \frac{7}{3}.$$

In contrast, we know that the gradient of a perpendicular line will be a negative reciprocal. Therefore, the gradient of our desired perpendicular line will be the negative reciprocal of $\frac{2}{3}$, which is $-\frac{3}{2}$. Again using the point-gradient form or solving for b manually yields:

$$y - (-1) = -\frac{3}{2}(x - 2) \quad \Rightarrow \quad y + 1 = -\frac{3}{2}x + 3 \quad \Rightarrow \quad y = -\frac{3}{2}x + 2.$$

4.4 Horizontal and vertical lines

Horizontal and vertical lines

- A **horizontal** line has gradient 0 and equation $y = b$.
- A **vertical** line has undefined gradient and equation $x = a$.
- The y-axis is $x = 0$, and the x-axis is $y = 0$.

Exercise 6

Write an equation for each line.

1. vertical line through $(-3, 2)$
2. line through $(-1, 2)$ and $(4, 2)$
3. line through $(0, 2)$ and $(0, -2)$
4. horizontal line through $(0, -4)$

5 Graphs of Linear Inequalities

Linear inequalities in two variables

A **linear inequality** in x and y can be written in one of the forms (where a and b are not both zero):

$$ax + by < c, \quad ax + by > c, \quad ax + by \leq c, \quad ax + by \geq c.$$

An ordered pair (x_1, y_1) is a **solution** if the inequality becomes a true statement when you substitute $x = x_1$ and $y = y_1$. For instance, $(3, 2)$ satisfies $x - y > 0$ because $3 - 2 > 0$ is true.

Graphing a linear inequality

To sketch the graph of a linear inequality:

1. Graph the **boundary line** by replacing the inequality sign with $=$.
2. Use a **dashed** boundary for $<$ or $>$, and a **solid** boundary for $\leq$ or $\geq$.
3. Pick a **test point** (often $(0, 0)$ if it is not on the boundary).
4. If the test point satisfies the inequality, shade the side containing it. Otherwise shade the other side.

The boundary line divides the coordinate chart into two regions (two **half-planes**). In one half-plane, **all** points satisfy the inequality, and in the other half-plane, **no** points satisfy it. So it is enough to test one point.

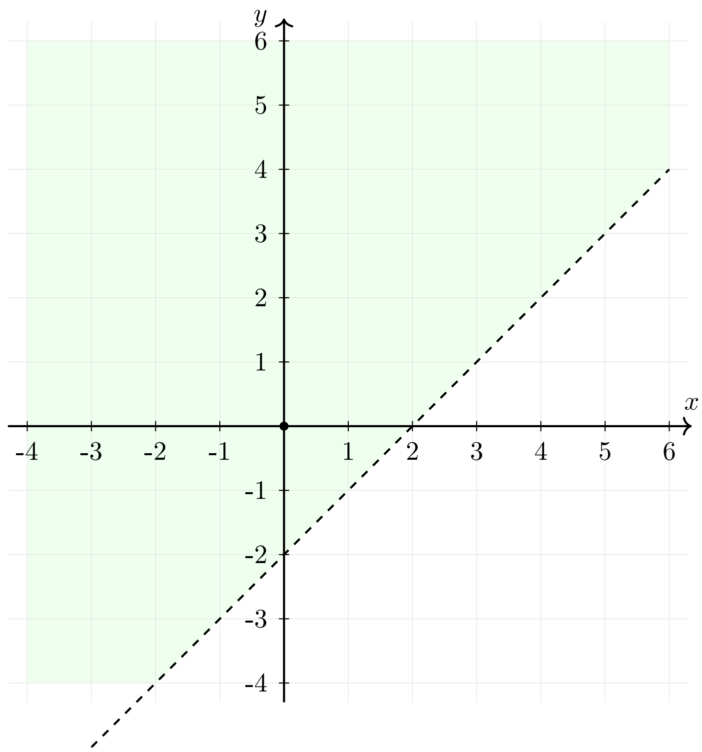
As an example, we will sketch the graph of the inequality $x - y < 2$. According to the above, we should first plot the line $x - y = 2$, which is equivalent to $y = x - 2$. Because the inequality is $<$, the boundary line is dashed. To decide which side to shade, we will use the test point $(0, 0)$. Substitution gives:

$$0 - 0 < 2 \quad \text{which is true.}$$

So, we shade the region containing the origin. In gradient-intercept language:

$$x - y < 2 \quad \Leftrightarrow \quad y > x - 2,$$

so we shade **above** the line $y = x - 2$. The graph of this linear inequality is depicted below.



Exercise 7

Use gradient-intercept form to sketch $5x + 4y \leq 12$.

Exercise 8

For each inequality, decide whether the point is a solution.

1. $x + y > 1$ at $(0, 0)$
2. $2x - y \leq 0$ at $(2, 3)$
3. $x \geq 4$ at $(3, -10)$

Solutions to the Exercises

Exercise 1

The line passes through $(0, 0)$ and $(1, -1)$, so

$$m = \frac{-1 - 0}{1 - 0} = -1.$$

Exercise 2

1. Through $(2, 3)$ and $(6, 7)$:
$$m = \frac{7-3}{6-2} = \frac{4}{4} = 1.$$
2. Through $(-1, 4)$ and $(3, 4)$:
$$m = \frac{4-4}{3-(-1)} = \frac{0}{4} = 0.$$
So the line is horizontal.
3. Through $(5, -2)$ and $(5, 1)$: the x-values are equal, so the line is vertical and the gradient is **undefined**.

Exercise 3

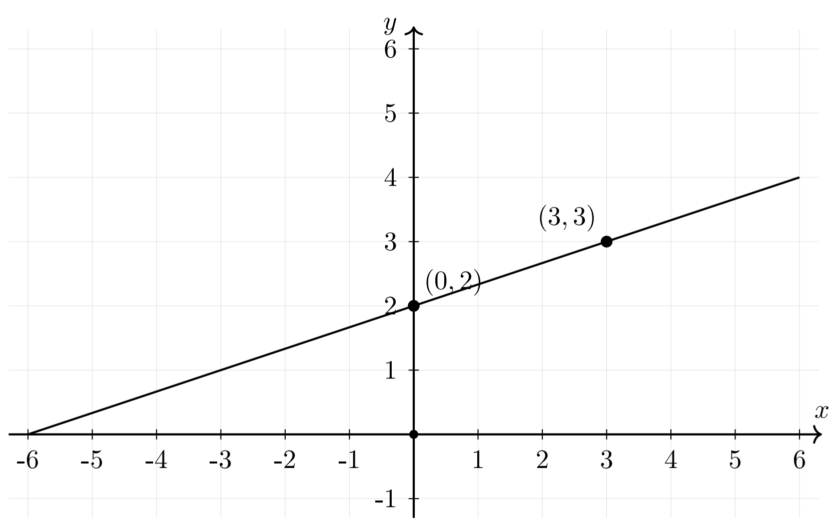
The ladder has endpoints $(0, 12)$ and $(5, 0)$. Therefore,
$$m = \frac{0-12}{5-0} = -\frac{12}{5}.$$

Exercise 4

First rewrite in gradient-intercept form:
$$x - 3y = -6 \quad \Rightarrow \quad -3y = -x - 6 \quad \Rightarrow \quad y = \frac{1}{3}x + 2.$$

So the gradient is $m = \frac{1}{3}$ and the y-intercept is $(0, 2)$.

To sketch: plot $(0, 2)$, then use rise-over-run. Since $m = \frac{1}{3}$, move 3 to the right and 1 up to reach $(3, 3)$, then draw the line through these points.



Exercise 5

1. $y = -3x - 2$ has gradient $m_1 = -3$. The line $y = \frac{1}{3}x + 1$ has gradient $m_2 = \frac{1}{3}$. Since $m_1 m_2 = -3 \cdot \frac{1}{3} = -1$, the lines are **perpendicular**.
2. Both lines $y = \frac{1}{2}x + 1$ and $y = \frac{1}{2}x - 1$ have gradient $\frac{1}{2}$, so they are **parallel**.

Exercise 6

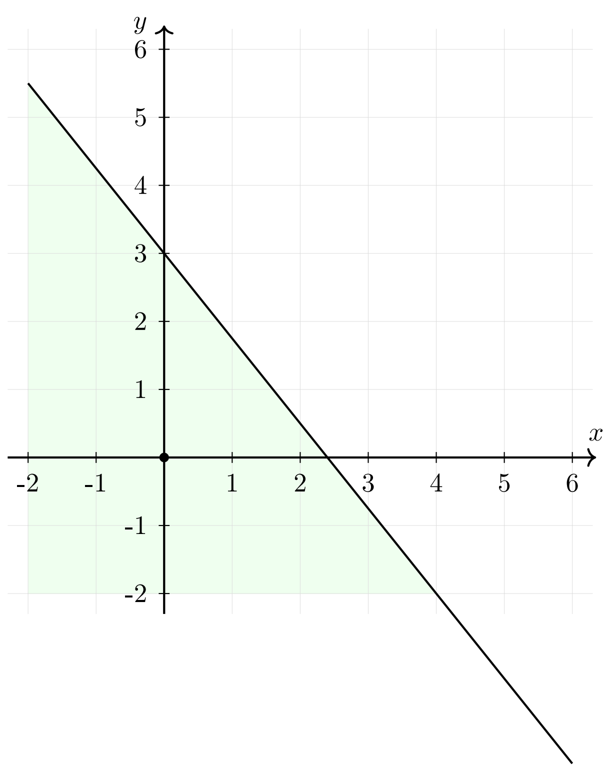
1. Vertical line through $(-3, 2)$: $x = -3$.
2. Through $(-1, 2)$ and $(4, 2)$ (same y-value): $y = 2$.
3. Through $(0, 2)$ and $(0, -2)$ (same x-value): $x = 0$.
4. Horizontal line through $(0, -4)$: $y = -4$.

Exercise 7

Rewrite in gradient-intercept form:

$$5x + 4y \leq 12 \quad \Rightarrow \quad 4y \leq -5x + 12 \quad \Rightarrow \quad y \leq -\frac{5}{4}x + 3.$$

So the boundary is the line $y = -\frac{5}{4}x + 3$, which is **solid** (because $\leq$), and we shade **on or below** the line.



(Quick check with $(0, 0)$: $5(0) + 4(0) \leq 12$ is true, so the shaded side should contain the origin, which it does.)

Exercise 8

1. At $(0, 0)$: $0 + 0 > 1$ is false, so $(0, 0)$ is **not** a solution.
2. At $(2, 3)$: $2(2) - 3 = 1 \leq 0$ is false, so $(2, 3)$ is **not** a solution.
3. At $(3, -10)$: $3 \geq 4$ is false, so $(3, -10)$ is **not** a solution.