MAT140 — Lecture 7 Handout

Exponents and Polynomials

In the previous lectures we have studied linear equations from an algebraic and geometric perspective. In this lecture, we will move on to a new object: polynomials. We will spend the next four lectures discussing these, before moving on to more advanced material.

So far, our guiding principle has been that mathematics behaves a lot like a language: there are symbols that can be used to express things, and these symbols have an interpretation in terms of geometry. In the language of mathematics, we have seen:

Language	**Mathematics**
Alphabet	Symbols
Words	Algebraic Expressions
Sentences	equations / inequalities
Grammar	Rules (properties)

Linear equations can be expressed and studied using only the arithmetic operations $+, -, \times$ and $\div$. In this lecture, we will explore algebraic expressions that contain exponents. Specifically, we:

- introduce **exponents** as a new piece of algebraic vocabulary, and learn the key rules for simplifying exponent expressions,
- learn **scientific notation** for expressing very large and very small numbers, and
- introduce **polynomials**, together with how to add, subtract, and multiply them.

1 Exponents and Their Rules

1.1 Exponent Notation

Recall that exponents are simply a compact way to write repeated multiplication. For example,

$$3^2 = 3 \times 3, \qquad 2^3 = 2 \times 2 \times 2, \qquad y^3 = y \cdot y \cdot y.$$

In general, if n is a positive integer, then x^n means multiplying x by itself n times.

Exponent notation

For a positive integer n,

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}.$$

The number x is called the **base**, and n is called the **exponent**.

1.2 Rules of Exponents

The following rules let us simplify expressions involving exponents.

Rules of Exponents

Let a and b be nonzero real numbers, and let m, n be positive integers. Then:

$$a^m \cdot a^n = a^{m+n}, \qquad \frac{a^m}{a^n} = a^{m-n} \quad (m > n),$$

$$(ab)^n = a^n b^n, \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n},$$
$$(a^m)^n = a^{mn}.$$

At first glance, it may not be immediately clear why these happen to be the rules of exponents. In fact, all of the rules above can be derived from the definition of exponentiation.

As an example, we will demonstrate that $a^m \cdot a^n = a^{m+n}$. We start by writing the left hand side using the definition of exponentiation:

$$(a^m)(a^n) = \underbrace{(a \cdot a \cdot \ldots \cdot a)}_{m \text{ factors}} \cdot \underbrace{(a \cdot a \cdot \ldots \cdot a)}_{n \text{ factors}}.$$

In total we have m factors of a, followed by another n factors right afterwards. This means that there are $(m + n)$ factors of a in total, and therefore

$$(a^m)(a^n) = \underbrace{(a \cdot a \cdot \ldots \cdot a)}_{(m+n) \text{ factors}} = a^{m+n}.$$

As another example, we will derive the rule $\frac{a^m}{a^n} = a^{m-n}$ when $m > n$. We start by using the definition of exponentiation to write out the fraction:

$$\frac{a^m}{a^n} = \frac{\overbrace{a \cdot a \cdot \ldots \cdot a}^{m \text{ factors}}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}}.$$

Since $m > n$, we can divide up the m factors in the numerator into two pieces: the first n factors, followed by the remaining $(m - n)$ factors. Then the first n factors cancel with the denominator, leaving only the $(m - n)$ factors behind:

$$\frac{a^m}{a^n} = \frac{\overbrace{(a \cdot a \cdot \ldots \cdot a)}^{n \text{ factors}}\overbrace{(a \cdot a \cdot \ldots \cdot a)}^{(m-n) \text{ factors}}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}} = \frac{\overbrace{(\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a})}^{n \text{ factors}}\overbrace{(a \cdot a \cdot \ldots \cdot a)}^{(m-n) \text{ factors}}}{\underbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}_{n \text{ factors}}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{(m-n) \text{ factors}} = a^{m-n}.$$

The other rules can be obtained in a similar way.

The rules above can also be used to simplify expressions that involve exponents of variables. For example:

- $(x^2y^4)(3x) = 3x^3y^4$.
- $-2(y^2)^3 = -2y^6$.
- $(-2y^2)^3 = -8y^6$.

Exercise 1

Simplify each expression.

(a) $(3x^2)(-5x)^3$

(b) $\dfrac{14a^5b^3}{7a^2b^2}$

(c) $\left(\dfrac{x^2}{2y}\right)^3$

(d) $\dfrac{x^n y^{3n}}{x^2y^4}$

1.3 Zero and Negative Exponents

In the previous section, we made the assumption that the exponents m and n were positive. As a matter of fact, we can also consider zero, or even *negative* exponents. The simplest example of this phenomenon comes from the multiplicative inverse property: for any nonzero real number a there is a unique number a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Of course, this number is simply the reciprocal of a:

$$a^{-1} = \frac{1}{a}.$$

We need to assume that a is nonzero, since the reciprocal of 0 divides by zero, which is not defined.

Using the fact that $a^{-1} = \frac{1}{a}$, we can obtain the following rules of exponents.

Zero and Negative Exponents

For any nonzero real number a and any positive integer m:

$$a^0 = 1, \qquad a^{-m} = \frac{1}{a^m}.$$

More generally, if $a, b \neq 0$, then

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.$$

Again, these rules may appear unusual at first. But they are easily obtained from the definition of exponents. For example, we can repeat the proof of $\frac{a^m}{a^n} = a^{m-n}$, this time taking $m = n$ to prove that $a^0 = 1$:

$$a^0 = a^{m-m} = \frac{a^m}{a^m} = \frac{\overbrace{a \cdot a \cdot \ldots \cdot a}^{m \text{ factors}}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}}} = \frac{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}} = 1.$$

Then we also have that

$$a^{-m} = a^{0-m} = \frac{a^0}{a^m} = \frac{1}{a^m}.$$

Exercise 2

Evaluate.

(a) 5^0
(b) 2^{-2}
(c) $\left(\frac{1}{2}\right)^{-2}$

2 Scientific Notation

Very large numbers are often hard to read if we write them out with all of their zeros. For example, consider the number

$$33{,}000{,}000{,}000{,}000{,}000{,}000.$$

As we can see, our usual number system starts to lose its convenience, and it becomes difficult to read properly. In order to solve this issue, in practice we often use scientific notation. This notation simplifies numbers containing many zeros into a shorter form, so that they become easier to read.

Scientific notation

A number is written in **scientific notation** if it is written in the form

$$a \times 10^n,$$

where n is an integer and a is a number satisfying $1 \leq |a| < 10$.

As an example, the number above can be rewritten in scientific notation as

$$33{,}000{,}000{,}000{,}000{,}000{,}000 = 3.3 \times 10^{19}.$$

Multiplication by 10 moves a number's decimal point by one place to the right, e.g. $7.2 \times 10 = 72$. So, the expression 3.3×10^{19} is saying that we ought to move the decimal point in the number 3.3 nineteen times to the right in order to get the other expression.

Conversely, division by 10 moves decimal places over to the left, e.g. $7.2 \div 10 = 0.72$. Therefore we may use repeated division by 10 to represent very small numbers using scientific notation:

$$6.84 \times 10^{-5} = 0.0000684.$$

Exercise 3

Write each number in scientific notation.

(a) 0.0072
(b) 937,200,000.0

3 Adding and Subtracting Polynomials

3.1 Defining a Polynomial

Polynomials are expressions that are built from numbers, variables, and nonnegative integer exponents. They are one of the most important families of expressions in algebra, and for this reason we will pay them a lot of attention throughout the next few lectures. We start with their definition.

Polynomials and terminology

A **polynomial** in x is any expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$ and the exponents are nonnegative integers.

- The **degree** of the polynomial is n.
- The **leading term** is $a_n x^n$.
- The **leading coefficient** is a_n.
- The **constant term** is a_0.
- In **standard form**, terms are written in descending order of exponent.

Polynomials with 1, 2, or 3 terms are called **monomials**, **binomials**, or **trinomials**, respectively.

As a matter of fact, we have already seen polynomials: linear equations are polynomials of degree 1:

$$mx + b \qquad \text{has} \qquad a_1 = m \text{ and } a_0 = b.$$

The table below has some examples of polynomials in standard form:

Polynomial	**Standard form**	**Degree**
$5x^2 - 2x^7 + 4 - 2x$	$-2x^7 + 5x^2 - 2x + 4$	7
$16 + x^2$	$x^2 + 16$	2
12	12	0

3.2 Adding Polynomials

Polynomials can be added in a straightforward way: to add two polynomials, we simply collect their like terms together. This is done by first inspecting the powers of the variable, and by adding the coefficients of the terms with the same powers. As an example, consider the following:

$$(2x^3 + x^2 - 5) + (x^2 + x + 6) = 2x^3 + (1 + 1)x^2 + x + (-5 + 6) = 2x^3 + 2x^2 + x + 1.$$

This method can also be applied to sums of more than two polynomials, for example:

$$(3x^2 + 2x + 4) + (3x^2 - 6x + 3) + (-x^2 + 2x - 4) = 5x^2 - 2x + 3.$$

Exercise 4

Add the following polynomials, and simplify your answer as much as possible.

(a) $(x^2 + x + 2) + (3x^3 + 2x^2 + x)$
(b) $(x^7 + 3) + (x^2 - 3) + (-x^2 + x^5)$

3.3 Subtracting Polynomials

Standard subtraction is really just another form of addition. For example, to subtract 2 from 3, we simply add the additive inverse of 2 to 3: $3 - 2 = 3 + (-2) = 1$. Similarly, we may subtract one polynomial from another by reversing signs and performing an addition.

Subtracting a Polynomial

Let $P(x) = a_n x^n + \ldots + a_1 x + a_0$ and $Q(x) = b_m x^m + \ldots + b_1 x + b_0$ be two polynomials. To subtract $Q(x)$ from $P(x)$, rewrite

$$P(x) - Q(x) = P(x) + \big(- Q(x) \big),$$

then distribute the negative sign into $Q(x)$, and perform the addition.

As an example of polynomial subtraction, we will subtract $(x^3 + 2x^2 - x - 4)$ from $(3x^3 - 5x^2 + 3)$:

$$\begin{aligned}(3x^3 - 5x^2 + 3) - (x^3 + 2x^2 - x - 4) &= (3x^3 - 5x^2 + 3) + (-x^3 - 2x^2 + x + 4) \\ &= 2x^3 - 7x^2 + x + 7.\end{aligned}$$

Common Mistake

When subtracting, the negative sign must be distributed to **every** term in the polynomial being subtracted. This means that we reverse the sign of *every* term in the polynomial. For example:

$$-\big(x^2 + 3x - 2\big) = -x^2 - 3x + 2, \quad \text{not} \quad -x^2 + 3x - 2.$$

Exercise 5

Subtract $(x^4 + x^3 - 2x + 1)$ from $(4x^3 - x^2 + 3x + 2)$.

4 Multiplying Polynomials

4.1 Multiplying by a Monomial

Multiplication of polynomials is more complicated than addition or subtraction. To make things easier to understand, we will start with the simplest cases and then build up the difficulty slowly.

The simplest situation is when we multiply a polynomial by a *monomial* – a polynomial with a single term. In this case, we can use the distributive property and multiply the monomial by each term in the polynomial. For example:

$$x(2x+5) = 2x^2 + 5x.$$

The distributive property also holds for polynomials with more terms. For example:

$$x(x^3 + 2x^2 + 4x - 2) = x^4 + 2x^3 + 4x^2 - 2x.$$

Exercise 6

Apply the distributive property to multiply the following polynomials.

(a) $(3x - 7)(-2x)$
(b) $3x^2(5x - x^3 + 2)$
(c) $(-x)(2x^2 - 3x)$

4.2 Multiplying Two Binomials

Next, we will consider the case that we multiply two binomials together, that is, two polynomials with two terms each. In this case, we can apply the distributive property twice to expand out the brackets. For example:

$$\begin{aligned}(3x-2)(5x+7) &= (3x-2)(5x) + (3x-2)(7) \\ &= (3x)(5x) + (-2)(5x) + (3x)(7) + (-2)(7) \\ &= 15x^2 - 10x + 21x - 14 \\ &= 15x^2 + 11x - 14.\end{aligned}$$

There is also a technique that allows us to multiply binomials in fewer steps.

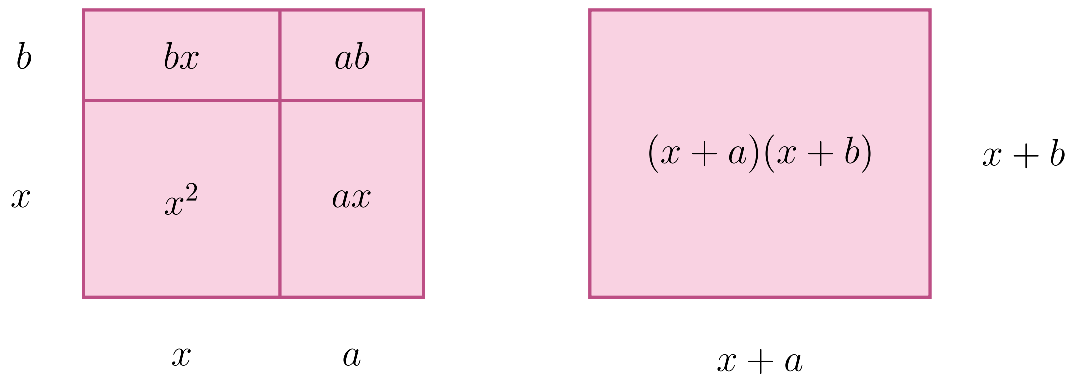
The FOIL Method

For two binomials $(A+B)(C+D)$:

$$(A+B)(C+D) = AC + AD + BC + BD.$$

A common mnemonic is **FOIL**: First, Outside, Inside, Last.

The FOIL method can also be understood geometrically. For convenience, suppose that we have a pair of simple binomials $x + a$ and $x + b$ that we would like to multiply together. We know that multiplication of positive numbers gives the area of a rectangle, so we may draw the expressions $(x+a)(x+b)$ and $x^2+(a+b)x+ab$ as follows.



As you can see, the left hand rectangle has area given by the sum of x^2, ax, bx and ab. However, the sum of these terms gives an overall rectangle whose area is precisely $(x+a)(x+b)$.

Exercise 7

Expand and simplify.

(a) $(x-1)(x+5)$
(b) $(2x+3)(x-2)$
(c) $(4x+5)^2$
(d) $(3x^2-2)(4x+7)-(4x)^2$

4.3 Other Multiplications

Using the principles above, we can also take products of polynomials that are more complicated. Consider first the product

$$(x-4)(x^2-4x+2).$$

Here we have to distribute by multiplying all possible pairs of terms coming from the two polynomials. Again, this can be considered as two applications of the distributive property. For example, consider the product:

$$\begin{aligned}(x-4)(x^2-4x+2) &= x^3-4x^2-4x^2+16x+2x-8\\ &= x^3-8x^2+18x-8.\end{aligned}$$

We can also use polynomial multiplication to compute exponents of polynomials. For example, the expression $(x-3)^3$ can be computed using two steps of binomial multiplication:

$$\begin{aligned}(x-3)^3 &= (x-3)(x-3)(x-3) \\ &= (x-3)\left((x-3)(x-3)\right) \\ &= (x-3)(x^2-6x+9) \\ &= x^3-3x^2-6x^2+18x+9x-27 \\ &= x^3-9x^2+27x-27.\end{aligned}$$

Exercise 8

Compute the following.

1. $(2x^2-7x+1)(4x+3)$
2. $(x+2)^3$

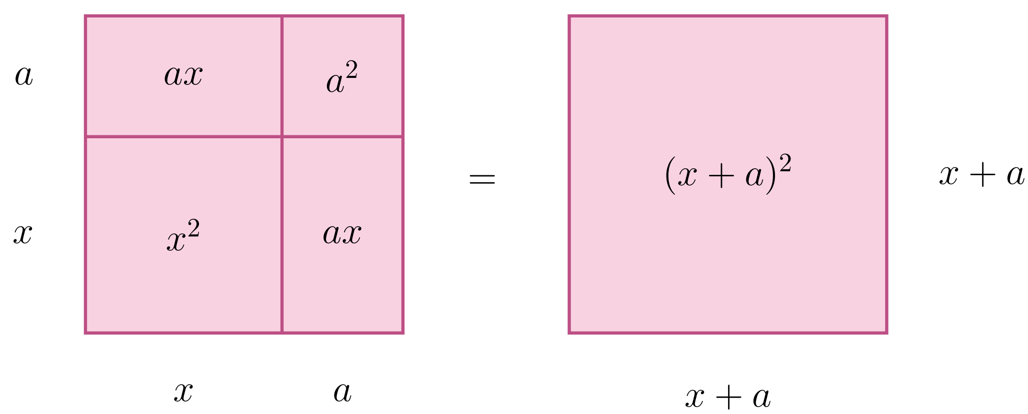
4.4 Special Products

Some binomial products appear so often that it is worth memorizing their patterns. This will be particularly useful for *factorization*, which we will discuss later on in the course.

Square of a binomial

$$(x+a)^2 = x^2+2ax+a^2, \qquad (x-a)^2 = x^2-2ax+a^2.$$

The square of a binomial can be seen visually as follows.



As you can see, the left hand square has area given by the sum of x^2, ax, ax and a^2. Therefore the total area is

$$x^2+ax+ax+a^2 = x^2+2ax+a^2,$$

which matches the area of the right hand square, whose side length is $x+a$. This gives the equality

$$(x+a)^2 = x^2+2ax+a^2.$$

Difference of squares

$$(x + a)(x - a) = x^2 - a^2.$$

Exercise 9 (special products)

Expand and simplify.

(a) $(5x - 6)(5x + 6)$
(b) $(3x + 7)^2$
(c) $(4x + 9)^2$
(d) $(6 - 5x^2)^2$

Solutions to the Exercises

Exercise 1

(a)
$$(3x^2)(-5x)^3 = 3x^2 \cdot (-125x^3) = -375x^5.$$

(b)
$$\frac{14a^5b^3}{7a^2b^2} = 2a^{5-2}b^{3-2} = 2a^3b.$$

(c)
$$\left(\frac{x^2}{2y}\right)^3 = \frac{(x^2)^3}{(2y)^3} = \frac{x^6}{8y^3}.$$

(d)
$$\frac{x^n y^{3n}}{x^2y^4} = x^{n-2}y^{3n-4}.$$

Exercise 2

(a) $5^0 = 1$.
(b) $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$.
(c) $\left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^2 = 4$.

Exercise 3

(a)
$$0.0072 = 7.2 \times 10^{-3}.$$

(b)
$$937{,}200{,}000.0 = 9.372 \times 10^8.$$

Exercise 4

(a)

$$(x^2 + x + 2) + (3x^3 + 2x^2 + x) = 3x^3 + (x^2 + 2x^2) + (x + x) + 2 = 3x^3 + 3x^2 + 2x + 2.$$

(b)

$$(x^7 + 3) + (x^2 - 3) + (-x^2 + x^5) = x^7 + x^5 + (x^2 - x^2) + (3 - 3) = x^7 + x^5.$$

Exercise 5

$$\begin{aligned}(4x^3 - x^2 + 3x + 2) - (x^4 + x^3 - 2x + 1) &= 4x^3 - x^2 + 3x + 2 - x^4 - x^3 + 2x - 1 \\ &= -x^4 + 3x^3 - x^2 + 5x + 1.\end{aligned}$$

Exercise 6

(a)

$$(3x - 7)(-2x) = (3x)(-2x) + (-7)(-2x) = -6x^2 + 14x.$$

(b)

$$3x^2(5x - x^3 + 2) = 15x^3 - 3x^5 + 6x^2 = -3x^5 + 15x^3 + 6x^2.$$

(c)

$$(-x)(2x^2 - 3x) = -2x^3 + 3x^2.$$

Exercise 7

(a)

$$(x - 1)(x + 5) = x^2 + 5x - x - 5 = x^2 + 4x - 5.$$

(b)

$$(2x + 3)(x - 2) = 2x^2 - 4x + 3x - 6 = 2x^2 - x - 6.$$

(c)

$$(4x + 5)^2 = (4x + 5)(4x + 5) = 16x^2 + 20x + 20x + 25 = 16x^2 + 40x + 25.$$

(d)

$$(3x^2 - 2)(4x + 7) - (4x)^2.$$

First expand:

$$(3x^2 - 2)(4x + 7) = 12x^3 + 21x^2 - 8x - 14.$$

Then subtract $(4x)^2 = 16x^2$:

$$12x^3 + 21x^2 - 8x - 14 - 16x^2 = 12x^3 + 5x^2 - 8x - 14.$$

Exercise 8

1.

$$(2x^2 - 7x + 1)(4x + 3) = 8x^3 + 6x^2 - 28x^2 - 21x + 4x + 3 = 8x^3 - 22x^2 - 17x + 3.$$

2.

$$(x + 2)^3 = (x + 2)^2(x + 2) = (x^2 + 4x + 4)(x + 2) = x^3 + 6x^2 + 12x + 8.$$

Exercise 9

(a)
$$(5x-6)(5x+6) = (5x)^2 - 6^2 = 25x^2 - 36.$$

(b)
$$(3x+7)^2 = 9x^2 + 42x + 49.$$

(c)
$$(4x+9)^2 = 16x^2 + 72x + 81.$$

(d)
$$(6-5x^2)^2 = 6^2 - 2(6)(5x^2) + (5x^2)^2 = 36 - 60x^2 + 25x^4.$$