MAT140 — Lecture 8 Handout

Polynomial Division

In Lecture 7 we completed most of the arithmetic of polynomials: we learned how to add, subtract, and multiply polynomials. In this lecture, we complete the picture by learning how to **divide one polynomial by another**.

A useful way to think about division is that it is often an *algorithm*: you follow a fixed set of steps, and (if you do the steps correctly) you will get the right answer every time. Polynomial long division is exactly this kind of process. So, at the very least, students should be able to understand how to apply the algorithm in order to produce correct results.

Alongside this, ideally students should be able to understand the main processes of division: in order to divide one thing by another, we are trying to "invert" multiplication, by asking "how many times does this thing fit into that thing". We can solve a problem like this by progressively improving our educated guesses, in a way that will eventually lead to the true result. At the heart of polynomial division is *division itself*, and therefore students should understand *why* this algorithm works.

Today we will:

1. Review the simplest case of polynomial division, in which we divide by a monomial (a polynomial with a single term).
2. Review the general idea of division and use our observations to divide polynomials from first principles.
3. Present the *long division algorithm* for polynomial division, which is essentially a nice, visual way to organise the division of polynomials.

1 Dividing a Polynomial by a Monomial

We start with the simplest case of polynomial division: dividing a polynomial by a monomial. A monomial is just a polynomial with a single term, which means that to divide a polynomial we can simply use basic rules of fractions and break up the division into a collection of sums.

Dividing by a Monomial

If $m(x)$ is a monomial and $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial, then

$$\begin{aligned}\frac{P(x)}{m(x)} &= \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{m(x)} \\ &= \frac{a_n x^n}{m(x)} + \frac{a_{n-1}x^{n-1}}{m(x)} + \cdots + \frac{a_1 x}{m(x)} + \frac{a_0}{m(x)}.\end{aligned}$$

Since monomials are only ever of the form bx^m, we can compute the division of a polynomial by a monomial by simply appealing to the exponent rules of last lecture – specifically, the rule:

$$\frac{x^n}{x^m} = x^{n-m}$$

for all positive m, n and $x \neq 0$.

As an example, suppose that we want to divide the polynomial $4x^2 + 2x$ by the monomial x. Using the scheme above, we have:

$$\frac{4x^2 + 2x}{x} = \frac{4x^2}{x} + \frac{2x}{x} = 4x + 2.$$

As a more complicated example, let's divide the polynomial $4x^4 + 3x^2 + x$ by the monomial $2x^2$. We can again compute the division term-by-term:

$$\frac{4x^4 + 3x^2 + x}{2x^2} = \frac{4x^4}{2x^2} + \frac{3x^2}{2x^2} + \frac{x}{2x^2} = 2x^2 + \frac{3}{2} + \frac{1}{2x}.$$

Exercise 1

Perform the division and simplify.

1. $\frac{x^5}{x}$
2. $\frac{x^3 + x}{x}$
3. $\frac{x^3 + 2x^2}{x}$
4. $\frac{7x^3 + 4x^2}{x^2}$

2 General Polynomial Division

2.1 Dividing Polynomials Evenly

Ordinary division is simply defined to be the inverse operation of multiplication, i.e. when we write something like $8 \div 2$, we are looking for a third number that multiplies by 2 to give the answer 8, we write:

$$8 \div 2 = 4 \quad \text{because} \quad 4 \times 2 = 8.$$

In the previous lecture, we noted that polynomials may always be multiplied together to yield a third, larger polynomial. For example, we could multiply the polynomials $(x + 1)$ and $(x + 2)$ together to yield:

$$(x + 1)(x + 2) = x^2 + 3x + 2.$$

Polynomial division is defined in a similar manner to ordinary division: it is the *inverse operation* of polynomial multiplication. This means that, conceptually at least, we may write something like:

$$\frac{x^2 + 3x + 2}{x + 1} = x + 2 \quad \text{because} \quad x^2 + 3x + 2 = (x + 1)(x + 2).$$

Later on in Section 3, we will describe in detail our "division algorithm" for polynomials. However, for now, we will spend some time studying a few properties of polynomial division.

Even Division of Polynomials

Given a pair of polynomials $P(x)$ and $D(x)$, we say that $D(x)$ *divides* $P(x)$ *evenly* if $D(x)$ is non-zero and there is a third polynomial $Q(x)$ such that

$$P(x) = Q(x) \cdot D(x)$$

We call $P(x)$ the **dividend**, $D(x)$ the **divisor**, and $Q(x)$ the **quotient**.

In the case of the division

$$\frac{x^2 + 3x + 2}{x + 1} = x + 2$$

the divisor is the polynomial $x+1$, the dividend is the polynomial x^2+3x+2, and the quotient is $x+2$.

Before getting to polynomial long division, it helps to recall some facts about ordinary division. Suppose first that we are trying to divide some large number by a smaller one.

For example, let's consider the division $60 \div 4$. One way to do this is to create a series of estimates, and then to refine these over time. We may ask something like "how many times does 4 go into 6?", for which the answer is one. Therefore, 4 must go into 60 at least 10 times. This is a nice first guess, but of course 4 goes into 60 more than 10 times, since 4×10 is only 40. We can directly calculate the "remainder", which I will call r:

$$60 = 10 \cdot 4 + r \quad \Rightarrow \quad r = 20.$$

Now, we simply need to find out how many times 4 goes into 20. As a matter of fact, $5 \cdot 4 = 20$, which is equivalent to writing $5 \cdot 4 = r$. We can use the equation above to rewrite 60 as:

$$60 = 10 \cdot 4 + r = 10 \cdot 4 + 5 \cdot 4 = (10 + 5) \cdot 4 = 15 \cdot 4.$$

If 60 equals 15×4, then equivalently $60 \div 4$ must equal 15.

In a similar spirit to the case of $60 \div 4$, we can divide polynomials by taking a reasonable first guess, and then figuring out how to refine our guess over time. Consider again the polynomial division:

$$\frac{x^2 + 3x + 2}{x + 1}.$$

In this case, we are trying to find a third polynomial $Q(x)$ such that

$$x^2 + 3x + 2 = Q(x)(x + 1).$$

Since there is an x^2 term in the dividend on the left-hand-side of the equation above, we observe that the polynomial $Q(x)$ *must* have an x term in it. Otherwise, there would be no way that $Q(x)$ could multiply with $(x + 1)$ to make an x^2 term. So, as our first guess, we will consider the polynomial $Q(x) = x$ and see what happens. If we perform the multiplication $x(x + 1)$, we now see that the right-hand-side of our equality becomes $x^2 + x$, which is clearly not equal:

$$x^2 + 3x + 2 \neq x^2 + x.$$

This is very much analogous to our observation that $60 \neq 40$ in our previous example. In that case, we calculated the remainder 20 by taking the difference $60 - 40 = 20$. In this case, we can now consider our "remainder" to be another polynomial $R(x)$ such that:

$$x^2 + 3x + 2 = x(x+1) + R(x).$$

We can do the same as before and perform a subtraction to find the value of $R(x)$. However, this time we need to perform a *polynomial* subtraction instead of an ordinary one:

$$R(x) = x^2 + 3x + 2 - (x^2 + x) = 2x + 2.$$

In exactly the same way that we then divided the remainder 20 by 4, we may now do the same here. In this case, our divisor is the polynomial $x + 1$, and we observe quite easily that:

$$2x + 2 = 2(x+1), \quad \text{therefore} \quad \frac{2x+2}{x+1} = 2.$$

We may conclude from this that the "remainder polynomial" $R(x) = 2x+2$ can be equally represented by the product $2(x+1)$. Putting this all together, we then have:

$$\begin{aligned} x^2 + 3x + 2 &= x(x+1) + 2(x+1) \\ &= (x+2)(x+1). \end{aligned}$$

2.2 Division with Remainders

Suppose now that we try to divide the number 25 by 4. In this case, we know that the answer is 6.25. Why? Because $25 = 6 \cdot 4 + 1$ and, once we divide both sides of this equation by 4, we get:

$$\frac{25}{4} = \frac{6 \cdot 4 + 1}{4} = \frac{6 \cdot 4}{4} + \frac{1}{4} = 6 + \frac{1}{4} = 6 + 0.25 = 6.25.$$

Here, we would say something like "4 goes into 25 six times, with remainder 1". This remainder of 1 then gets divided by 4 to give us the decimal expansion 0.25.

In the same way, polynomial division doesn't always come out perfectly, and we can sometimes obtain "remainder terms" like the 0.25 from before. In this case, we have to divide this extra remainder term by the divisor, and that gives us a *polynomial fraction*. The important details are summarized below.

Polynomial Division with Remainders

Given a pair of polynomials $P(x)$ and $D(x)$, we say that $D(x)$ *divides* $P(x)$ *with remainder* if $D(x)$ is non-zero and there are polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = Q(x) \cdot D(x) + R(x).$$

We require that $R(x)$ has **a smaller degree than** $D(x)$. We refer to this additional, smaller polynomial $R(x)$ as the *remainder*.

In the situation in which we have a remainder term, we simply express the overall division by

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}.$$

As an example, let's consider the polynomial division:

$$\frac{2x^2 - 1}{x + 2}.$$

We are ultimately looking for polynomials $Q(x)$ and $R(x)$ such that

$$2x^2 - 1 = Q(x)(x + 2) + R(x), \qquad \deg(R) < \deg(x + 2) = 1.$$

Observe first that since $\deg(R) < 1$, this means that we expect the polynomial $R(x)$ to be only a constant term.

Now, to compute this polynomial division we will first estimate the value of the quotient $Q(x)$ by intelligently picking some polynomial $Q_1(x)$. We construct $Q_1(x)$ by counting its required degree, and then we will refine it from there. We start by reminding ourselves that we are trying to find a polynomial $Q_1(x)$ such that:

$$2x^2 - 1 = Q_1(x)(x + 2).$$

In order for this to be the case, the top term in $Q_1(x)$ must multiply together with x to create the leading term $2x^2$ in P on the left hand side. Therefore, it *must* be the case that $Q_1(x)$ contains the term $2x$. However, it is likely that this first guess will not be correct, i.e. there will be some "remainder term", which I will write as $R_1(x)$, such that:

$$2x^2 - 1 = 2x(x + 2) + R_1(x).$$

To find the exact form of $R_1(x)$, we subtract $2x(x + 2)$ from $2x^2 - 1$ and see what remains:

$$R_1(x) = (2x^2 - 1) - 2x(x + 2) = 2x^2 - 1 - 2x^2 - 4x = -4x - 1.$$

We may equivalently write this as:

$$2x^2 - 1 = 2x(x + 2) - 4x - 1.$$

Now we turn our attention to the polynomial $R_1(x) = -4x - 1$. We want to express this "remainder term" as a product of something with $x + 2$ (in the same way that we expressed $20 = 5 \cdot 4$ in the example of Section 2.1). So, we divide $R_1(x)$ by $x + 2$ to see how many more times we can fit $x + 2$ into the overall polynomial $P(x)$. The division is:

$$-4x - 1 = Q_2(x)(x + 2) + R_2(x).$$

If R_2 has a smaller degree than the polynomial $x + 2$, then in order for this equation to be true, it *must* be the case that Q_2 contains a leading term of -4. So, let's assume that $Q_2 = -4$ and reverse-engineer the value of R_2:

$$-4x - 1 = -4(x + 2) + R_2(x) \quad \Rightarrow \quad R_2 = -4x - 1 - (-4)(x + 2) = -4x - 1 + 4x + 8 = 7.$$

Thus $R_2(x)$ is simply the constant term 7, which is a polynomial whose degree is less than that of the divisor $x + 2$. In other words: the polynomial R_2 is "smaller" than the divisor $x + 2$ (in the same way that the remainder 1 was smaller than 4 in our previous example), and therefore we stop our process here. Putting this all together, we have:

$$\begin{aligned} 2x^2 - 1 &= Q_1(x)(x+2) + R_1(x) \\ &= (2x)(x+2) + (-4x-1) \\ &= (2x)(x+2) + (-4)(x+2) + 7 \\ &= (2x-4)(x+2) + 7. \end{aligned}$$

This is of the form $P(x) = Q(x)D(x) + R(x)$, where here:

- the dividend $P(x) = 2x^2 - 1$,
- the divisor $D(x) = x + 2$,
- the quotient is $Q(x) = 2x - 4$ (which is also equal to the sum of our guesses $Q_1 + Q_2$), and
- the remainder term is $R(x) = 7$ (which is also equal to our final remainder R_2).

The final result of our polynomial division can be repackaged as:

$$\frac{2x^2 - 1}{x + 2} = 2x - 4 + \frac{7}{x + 2}.$$

In the previous example, we knew to stop our process once we reached the remainder term $R_2 = 7$. Why? Because this remainder has a degree strictly less than the divisor $x + 2$. This is an important rule that is worth emphasizing.

The Rule for Remainders

When dividing $P(x)$ by $D(x)$, the remainder $R(x)$ must satisfy

$$\deg(R) < \deg(D).$$

If the remainder had degree *at least* that of the divisor, then we could keep dividing.

3 The Polynomial Long Division Algorithm

3.1 Long Division for Integers

The polynomial long division algorithm is an adaptation of the same algorithm used for integers. So, we will briefly recall the process. As an example, we will compute $377 \div 13$ i.e. $13\overline{)377}$

$$\begin{array}{rrcrrr} & & & 0 & & \\ \cline{4-6} 1 & 3 &) & 3 & 7 & 7 \\ & & & 0 & & \\ \cline{3-4} \end{array} \xrightarrow{(1)} \begin{array}{rrcrrr} & & & 0 & & \\ \cline{4-6} 1 & 3 &) & 3 & 7 & 7 \\ & & & 0 & \downarrow & \\ \cline{3-5} & & & 3 & 7 & \end{array} \xrightarrow{(2)} \begin{array}{rrcrrr} & & & 0 & 2 & \\ \cline{4-6} 1 & 3 &) & 3 & 7 & 7 \\ & & & 0 & & \\ \cline{3-5} & & & 3 & 7 & \\ & & & 2 & 6 & \end{array} \xrightarrow{(3)} \begin{array}{rrcrrrr} & & & 0 & 2 & & \\ \cline{4-6} 1 & 3 &) & 3 & 7 & 7 & \\ & & & 0 & & \downarrow & \\ \cline{3-4} & & & 3 & 7 & \downarrow & \\ & & - & 2 & 6 & \downarrow & \\ \cline{3-5} & & & 1 & 1 & 7 & \end{array} \xrightarrow{(4)} \begin{array}{rrcrrrr} & & & 0 & 2 & 9 & \\ \cline{4-6} 1 & 3 &) & 3 & 7 & 7 & \\ & & & 0 & & \downarrow & \\ \cline{3-4} & & & 3 & 7 & \downarrow & \\ & & - & 2 & 6 & \downarrow & \\ \cline{3-6} & & & 1 & 1 & 7 & \\ & & & 1 & 1 & 7 & \\ \cline{3-6} & & & & & 0 & \end{array}$$

(1) 13 doesn't go into 3 (i.e. goes in 0 times), so we write zero in the hundreds column, and subtract $3 - 0$. The 7 from the tens column is brought down to make 37.

(2) 13 goes into 37 two times, i.e. $13 \times 2 = 26$, so we write the 2 in the tens column and 26 below the 37.

(3) Perform the subtraction: $37 - 26 = 11$ and bring the 7 from the units column down to make 117.

(4) 13 goes into 117 nine times, i.e. $13 \times 9 = 117$. So, write the 9 in the units column at the top, and we are done.

Exercise 2

Use long division to perform the following.

1. $514 \div 12$
2. $6230 \div 15$
3. $1234 \div 3$
4. $14302 \div 11$

3.2 Polynomial Long Division

Polynomial division is the same idea as integer division, except now "smaller" means **lower degree**. If we want to divide a polynomial $P(x)$ by another $D(x)$, then our task is really to write

$$P(x) = Q(x)\,D(x) + R(x),$$

where $Q(x)$ is the quotient, and $R(x)$ is the remainder, which has degree strictly less than $D(x)$. The polynomial long division algorithm is essentially a way to find $Q(x)$ by taking successive choices $Q_1(x), Q_2(x)$ and so on – just as we did in Section 2. In each case, we found that our "guess" for the correct quotient was incorrect, because there was a remainder term. So, we just took this remainder term and divided it by $D(x)$ and kept going until we obtained a remainder so small that we couldn't divide by $D(x)$ any more. This gives an overall process something like this:

$$P(x) = Q_1(x)D(x) + \underbrace{R_1(x)}$$

$$\downarrow$$

$$R_1(x) = Q_2(x)D(x) + \underbrace{R_2(x)}$$

$$\downarrow$$

$$R_2(x) = Q_3(x)D(x) + \underbrace{R_3(x)}$$

$$\downarrow$$

$$\vdots$$

This keeps producing remainders of smaller and smaller degrees, and therefore the process terminates after some finite number of steps.[1] The final quotient $Q(x)$ can be described by collecting together

[1] In fact, the process will take at most $\deg(P) - \deg(D) + 1$ steps.

all of the quotients of this process, and the final remainder term $R(x)$ is simply the value of the remainder term that eventually has degree smaller than $D(x)$. In symbols:

$$\begin{aligned} P(x) &= Q_1(x)D(x) + R_1(x) \\ &= Q_1(x)D(x) + \big(Q_2(x)D(x) + R_2(x)\big) \\ &= Q_1(x)D(x) + \big(Q_2(x)D(x) + (Q_3(x)D(x) + R_3(x))\big) \\ &\vdots \text{ (finitely-many steps } m) \\ &= Q_1(x)D(x) + Q_2(x)D(x) + \cdots + Q_m(x)D(x) + R_m(x) \\ &= \underbrace{\Big(Q_1(x) + Q_2(x) + \cdots + Q_m(x)\Big)}_{Q(x)} D(x) \;+\; \underbrace{R_m(x)}_{R(x)}. \end{aligned}$$

This can all be summarised as follows.

Polynomial long division (algorithm)

To divide $P(x)$ by $D(x)$:

1. Put both polynomials in **standard form** (descending powers).
2. Divide the **leading term** of $P(x)$ by the **leading term** of $D(x)$ to get the next term Q_i of $Q(x)$.
3. Multiply $D(x)$ by that new term and subtract from $P(x)$.
4. Repeat on the new polynomial until the remainder has degree smaller than the divisor.

The polynomial long division algorithm is simply a nice way to visually arrange the repeated derivations of the Q's and R's of the previous section. The Q terms are found one-by-one by dividing the leading term in the previous R by the leading term in the divisor $D(x)$. We do this, then multiply out $D(x)$ by the newly-found Q in order to find the next R polynomial. Repeating this process (in a manner very similar to the long division algorithm for integers) will progressively create the final quotient term Q.

$$\begin{array}{rl} & \quad Q_1(x) + Q_2(x) + \quad \cdots \quad + Q_m(x) \\ D(x) & \big) \quad \longleftarrow P(x) \longrightarrow \\ - & \underline{Q_1(x)D(x)} \\ & \qquad R_1(x) \\ & - \quad \underline{Q_2(x)D(x)} \\ & \qquad\qquad\qquad R_2(x) \\ & \qquad\qquad\qquad\qquad \ddots \\ & \qquad\qquad\qquad\qquad\quad R_{m-1}(x) \\ & \qquad\qquad\qquad - \quad \underline{Q_m(x)D(x)} \\ & \qquad\qquad\qquad\qquad\quad R_m(x) \end{array}$$

3.3 Worked Examples

We will now thoroughly review the long division algorithm step-by-step. In Section 2 we saw two instances of polynomial division, namely

$$\frac{x^2+3x+2}{x+1} = x+2 \quad \text{and} \quad \frac{2x^2-1}{x+2} = 2x-4+\frac{7}{x+2}.$$

To begin with, we will demonstrate how to derive these results using the polynomial long-division algorithm. Then, we will look at a new example.

Example 1

We will compute the division $\dfrac{x^2+3x+2}{x+1}$, which can be written equivalently in the form

$$x+1\overline{)x^2+3x+2}$$

In this case, the dividend is the polynomial x^2+3x+2, and the divisor is the polynomial $x+1$. We start by identifying the leading terms in each polynomial, and then performing a division. In this case, the leading term of the dividend is x^2 and the leading term of the divisor is x. We perform the division $x^2 \div x = x$ and write the solution in the answer row:

$$\begin{array}{r|lll} & x & & \\ \hline x+1) & x^2 & +3x & +2 \end{array}$$

Now, we multiply the entire divisor $x+1$ by the value x, and we write the answer in the row underneath the x^2+3x:

$$\begin{array}{r|lll} & x & & \\ \hline x+1) & x^2 & +3x & +2 \\ & x^2 & +x & \end{array}$$

From here, we do the following steps:

$$\begin{array}{rlll} & x & & \\ \cline{2-4} x+1) & x^2 & +3x & +2 \\ & x^2 & +x & \\ \cline{2-3} \end{array} \xrightarrow{(1)} \begin{array}{rlll} & x & & \\ \cline{2-4} x+1) & x^2 & +3x & +2 \\ - & x^2 & +x & \downarrow \\ \cline{2-3} & & 2x & +2 \end{array} \xrightarrow{(2)} \begin{array}{rlll} & x & +2 & \\ \cline{2-4} x+1) & x^2 & +3x & +2 \\ & x^2 & +x & \\ \cline{2-3} & & 2x & +2 \\ & & 2x & +2 \end{array} \xrightarrow{(3)} \begin{array}{rlll} & x & +2 & \\ \cline{2-4} x+1) & x^2 & +3x & +2 \\ & x^2 & +x & \\ \cline{2-3} & & 2x & +2 \\ - & & 2x & +2 \\ \cline{3-4} & & & 0 \end{array}$$

(1) We subtract $(x^2+3x+2)-(x^2+x) = 2x+2$. This removes the x^2 terms from the division and leaves us with a simpler polynomial $2x+2$. Here, we have moved down the constant term $+2$ and include it in our answer – the symbol $\downarrow$ indicates the constant term coming down.

(2) We again perform a division with the leading term in the divisor $x+1$. In this case, we divide: $2x \div x = 2$. Write $+2$ in the solution row at the top, and then multiply this with the whole polynomial $x+1$. We obtain: $2(x+1) = 2x+2$, and we write this answer in the bottom row.

(3) We subtract the two available rows: $(2x+2)-(2x+2) = 0$. The subtraction yields a zero, and therefore we are done.

Finally, we read off the polynomial in the solution row at the top of our array to conclude that:

$$\frac{x^2+3x+2}{x+1} = x+2.$$

We may double-check our solution by multiplying out by our divisor and computing the multiplication:

$$x^2+3x+2 = (x+2)(x+1) = x^2+2x+x+2 = x^2+3x+2,$$

and therefore our solution is correct.

Example 2

We will compute the division $\dfrac{2x^2-1}{x+2}$, which we can write as

$$x+2\overline{)\,2x^2+0x-1}$$

The only extra step here is to make sure to write the dividend in the standard form $2x^2+0x-1$, so that every power of x has its own "column". Now, we proceed as before by identifying the leading terms in the divisor and the dividend. The dividend starts with the term $2x^2$ and the divisor starts with the term x. So, we divide $2x^2 \div x = 2x$, and we write $2x$ in the answer row at the top:

$$\begin{array}{r|rrr} \multicolumn{1}{r}{} & 2x & & \\ \cline{2-4} x+2) & 2x^2 & +0x & -1 \end{array}$$

Next, we multiply the entire divisor by the $2x$ we just wrote:

$$2x(x+2) = 2x^2+4x.$$

We place this underneath the matching columns:

$$\begin{array}{r|rrr} \multicolumn{1}{r}{} & 2x & & \\ \cline{2-4} x+2) & 2x^2 & +0x & -1 \\ \multicolumn{1}{r}{} & 2x^2 & +4x & \end{array}$$

From here, we repeat the same process as in Example 1: we subtract to find the next remainder, bring down the next term in the dividend and then repeat the process until we naturally finish. The completed process is below.

$$\begin{array}{r|rrr} \multicolumn{1}{r}{} & 2x & & \\ \cline{2-4} x+2) & 2x^2 & +0x & -1 \\ \multicolumn{1}{r}{} & 2x^2 & +4x & \\ \cline{2-3} \end{array} \xrightarrow{(1)} \begin{array}{rr|rrr} & \multicolumn{1}{r}{} & 2x & & \\ \cline{3-5} & x+2) & 2x^2 & +0x & -1 \\ - & \multicolumn{1}{r}{} & 2x^2 & +4x & \downarrow \\ \cline{3-4} & \multicolumn{1}{r}{} & & -4x & -1 \end{array} \xrightarrow{(2)} \begin{array}{r|rrr} \multicolumn{1}{r}{} & 2x & -4 & \\ \cline{2-4} x+2) & 2x^2 & +0x & -1 \\ \multicolumn{1}{r}{} & 2x^2 & +4x & \\ \cline{2-3} \multicolumn{1}{r}{} & & -4x & -1 \\ \multicolumn{1}{r}{} & & -4x & -8 \end{array} \xrightarrow{(3)} \begin{array}{rr|rrr} & \multicolumn{1}{r}{} & 2x & -4 & \\ \cline{3-5} & x+2) & 2x^2 & +0x & -1 \\ & \multicolumn{1}{r}{} & 2x^2 & +4x & \\ \cline{2-4} & \multicolumn{1}{r}{} & & -4x & -1 \\ - & \multicolumn{1}{r}{} & & -4x & -8 \\ \cline{4-5} & \multicolumn{1}{r}{} & & & 7 \end{array}$$

(1) We subtract the first two rows: $(2x^2+0x-1)-(2x^2+4x) = -4x-1$, which gives us our new remainder. The $\downarrow$ is just reminding us that the constant term -1 comes straight down into the next line.

(2) We inspect the leading term of the new remainder that we have created, and divide it by the leading term of our divisor. Here, we have: $(-4x) \div x = -4$, and we write the solution -4 in the answer row at the top. Then, we multiply -4 by the full divisor: $-4(x+2) = -4x - 8$, and place the answer underneath.

(3) We subtract again: $(-4x - 1) - (-4x - 8) = 7$. Since 7 is a polynomial with degree 0, and the divisor $x + 2$ has degree 1, we are done.

The final expression $2x - 4$ written on the top row gives us the quotient, and in this case there is an extra remainder of 7. Putting this all together, our algorithm has told us that:

$$2x^2 - 1 = (2x-4)(x+2) + 7, \qquad \text{equivalently} \qquad \frac{2x^2-1}{x+2} = 2x - 4 + \frac{7}{x+2}.$$

Example 3

We will compute the division $\dfrac{x^2+2x+4}{x-1}$, written as

$$x - 1 \overline{\big)\, x^2 + 2x + 4}$$

Here the dividend is $x^2 + 2x + 4$ and the divisor is $x - 1$. We start the same way: compare leading terms. Since $x^2 \div x = x$ we start by writing x in the answer row:

$$\begin{array}{r|lll} & x & & \\ \hline x-1 & x^2 & +2x & +4 \end{array}$$

Now, we multiply the divisor by x

$$x(x-1) = x^2 - x,$$

and write that underneath the first two columns:

$$\begin{array}{r|lll} & x & & \\ \hline x-1 & x^2 & +2x & +4 \\ & x^2 & -x & \end{array}$$

Now we subtract, bring the constant down, and repeat the process again:

$$\begin{array}{r|lll} & x & & \\ \hline x-1 & x^2 & +2x & +4 \\ & x^2 & -x & \\ \hline \end{array} \xrightarrow{(1)} \begin{array}{rr|lll} & & x & & \\ \hline x-1 & & x^2 & +2x & +4 \\ & - & x^2 & -x & \downarrow \\ \hline & & & 3x & +4 \end{array} \xrightarrow{(2)} \begin{array}{r|lll} & x & +3 & \\ \hline x-1 & x^2 & +2x & +4 \\ & x^2 & -x & \\ \hline & & 3x & +4 \\ & & 3x & -3 \end{array} \xrightarrow{(3)} \begin{array}{rr|lll} & & x & +3 & \\ \hline x-1 & & x^2 & +2x & +4 \\ & & x^2 & -x & \\ \hline & & & 3x & +4 \\ & - & & 3x & -3 \\ \hline & & & & 7 \end{array}$$

(1) Subtract: $(x^2 + 2x + 4) - (x^2 - x) = 3x + 4$. The $\downarrow$ is indicating that the constant $+4$ drops into the new row.

(2) Divide leading terms again: here the leading term of the remainder $3x + 4$ is the term $3x$, so we perform the division $(3x) \div x = 3$. We write the solution $+3$ on the top row, and then multiply the full divisor $x - 1$ by this answer: $3(x-1) = 3x - 3$, and place it underneath.

(3) We now subtract: $(3x+4)-(3x-3)=7$. Since 7 has degree 0 and the divisor $x-1$ has degree 1, we can go no further, i.e. we stop here.

According to our algorithm, the quotient is $x+3$ and the remainder is 7:

$$x^2+2x+4=(x+3)(x-1)+7, \qquad \text{equivalently} \qquad \frac{x^2+2x+4}{x-1}=x+3+\frac{7}{x-1}.$$

We may always double-check our solution by multiplying everything out: $(x+3)(x-1)=x^2+2x-3$, and then adding 7 gives x^2+2x+4.

Exercise 3

Use the long division algorithm to compute the following. Give your answer as $\dfrac{P(x)}{D(x)}=Q(x)+\frac{R(x)}{D(x)}$.

1. $\dfrac{4x^2-3}{x+6}$
2. $\dfrac{2x^3-3x}{x+3}$
3. $\dfrac{2x^4-3x^2-x}{x+2}$
4. $\dfrac{x^3-3x^2-x}{x+1}$

Solutions to the Exercises

Exercise 1

1. $\dfrac{x^5}{x}=x^4$.
2. $\dfrac{x^3+x}{x}=x^2+1$.
3. $\dfrac{x^3+2x^2}{x}=x^2+2x$.
4. $\dfrac{7x^3+4x^2}{x^2}=7x+4$.

Exercise 2

1.

$$\begin{array}{r|rrr} & & 4 & 2 \\ \hline 1\ 2 & 5 & 1 & 4 \\ & 4 & 8 & \\ \cline{2-3} & & 3 & 4 \\ & & 2 & 4 \\ \cline{3-4} & & 1 & 0 \end{array}$$

So $514 \div 12 = 42$ remainder 10, i.e.

$$514 \div 12 = 42+\frac{10}{12}=42+\frac{5}{6}.$$

2.

$$
\begin{array}{rrlrrrr}
 & & & & 4 & 1 & 5 \\ \cline{4-7}
1 & 5 & \big) & 6 & 2 & 3 & 0 \\
 & & & 6 & 0 & & \\ \cline{4-5}
 & & & & 2 & 3 & \\
 & & & & 1 & 5 & \\ \cline{5-6}
 & & & & & 8 & 0 \\
 & & & & & 7 & 5 \\ \cline{6-7}
 & & & & & & 5
\end{array}
$$

So $6230 \div 15 = 415$ remainder 5, i.e.

$$6230 \div 15 = 415 + \frac{5}{15} = 415 + \frac{1}{3}.$$

3.

$$
\begin{array}{rlrrrr}
 & & & 4 & 1 & 1 \\ \cline{3-6}
3 & \big) & 1 & 2 & 3 & 4 \\
 & & 1 & 2 & & \\ \cline{3-4}
 & & & & 3 & \\
 & & & & 3 & \\ \cline{5-5}
 & & & & & 4 \\
 & & & & & 3 \\ \cline{6-6}
 & & & & & 1
\end{array}
$$

So $1234 \div 3 = 411$ remainder 1, i.e.

$$1234 \div 3 = 411 + \frac{1}{3}.$$

4.

$$
\begin{array}{rrlrrrrr}
 & & & & 1 & 3 & 0 & 0 \\ \cline{4-8}
1 & 1 & \big) & 1 & 4 & 3 & 0 & 2 \\
 & & & 1 & 1 & & & \\ \cline{4-5}
 & & & & 3 & 3 & & \\
 & & & & 3 & 3 & & \\ \cline{5-6}
 & & & & & & 0 & \\
 & & & & & & 0 & \\ \cline{7-7}
 & & & & & & & 2
\end{array}
$$

So $14302 \div 11 = 1300$ remainder 2, i.e.

$$14302 \div 11 = 1300 + \frac{2}{11}.$$

Exercise 3

1.

$$
\begin{array}{rlrrr}
 & & 4x & -\ 24 & \\ \cline{3-5}
x+6 & \big) & 4x^2 & +\ 0x & -\ 3 \\
- & & 4x^2 & +\ 24x & \\ \cline{3-4}
 & & & -\ 24x & -\ 3 \\
 & - & & -\ 24x & -\ 144 \\ \cline{4-5}
 & & & & 141
\end{array}
$$

Hence

$$\frac{4x^2-3}{x+6} = 4x - 24 + \frac{141}{x+6}.$$

2.

$$\begin{array}{rrrrrr}
 & & 2x^2 & -\,6x & +\,15 & \\
\cline{3-6}
x+3 & \big) & 2x^3 & +\,0x^2 & -\,3x & +\,0 \\
 & - & 2x^3 & +\,6x^2 & & \\
\cline{3-4}
 & & & -\,6x^2 & -\,3x & \\
 & & - & -\,6x^2 & -\,18x & \\
\cline{4-5}
 & & & & 15x & +\,0 \\
 & & & - & 15x & +\,45 \\
\cline{5-6}
 & & & & & -\,45
\end{array}$$

Hence

$$\frac{2x^3-3x}{x+3} = 2x^2 - 6x + 15 - \frac{45}{x+3}.$$

3.

$$\begin{array}{rrrrrrr}
 & & 2x^3 & -\,4x^2 & +\,5x & -\,11 & \\
\cline{3-7}
x+2 & \big) & 2x^4 & +\,0x^3 & -\,3x^2 & -\,x & +\,0 \\
 & - & 2x^4 & +\,4x^3 & & & \\
\cline{3-4}
 & & & -\,4x^3 & -\,3x^2 & & \\
 & & - & -\,4x^3 & -\,8x^2 & & \\
\cline{4-5}
 & & & & 5x^2 & -\,x & \\
 & & & - & 5x^2 & +\,10x & \\
\cline{5-6}
 & & & & & -\,11x & \\
 & & & & - & -\,11x & -\,22 \\
\cline{6-7}
 & & & & & & 22
\end{array}$$

Hence

$$\frac{2x^4-3x^2-x}{x+2} = 2x^3 - 4x^2 + 5x - 11 + \frac{22}{x+2}.$$

4.

$$\begin{array}{rrrrrr}
 & & x^2 & -\,4x & +\,3 & \\
\cline{3-6}
x+1 & \big) & x^3 & -\,3x^2 & -\,x & +\,0 \\
 & - & x^3 & +\,x^2 & & \\
\cline{3-4}
 & & & -\,4x^2 & -\,x & \\
 & & - & -\,4x^2 & -\,4x & \\
\cline{4-5}
 & & & & 3x & +\,0 \\
 & & & - & 3x & +\,3 \\
\cline{5-6}
 & & & & & -\,3
\end{array}$$

Hence

$$\frac{x^3-3x^2-x}{x+1} = x^2 - 4x + 3 - \frac{3}{x+1}.$$