MAT140 — Lecture 9 Handout
Factorization

In Lecture 8 we saw how to divide one polynomial by another. This completed a certain observation: polynomials *rhyme with numbers*, in the sense that they can be added, subtracted, multiplied, and divided in similar ways to that of numbers. Of course there were slight differences, but generally we can understand the set of polynomials as behaving similarly to the set of all integers $\{\ldots,-2,-1,0,1,2,\ldots\}$.

In this lecture we start talking about something that feels much more like the "number theory" of polynomials: **factorization**.

A good way to think about factorization is that it is the opposite procedure of **expanding brackets**. For example, we know how to expand

$$(x-1)(x+1)(x-2) \quad \longrightarrow \quad x^3-2x^2-x+2.$$

Factorization asks for the reverse:

$$x^3-2x^2-x+2 \quad \longrightarrow \quad (x-1)(x+1)(x-2).$$

Just as integers can be broken into prime factors, polynomials can often be broken into simpler "building blocks" as products.

Today we will:

1. Learn how to factor polynomials by taking out a **greatest common monomial factor**.
2. Learn how to factor **trinomials** of the form x^2+bx+c (and also trinomials in two variables).
3. Learn how to **factor completely**, and how to factor trinomials of the form ax^2+bx+c when $a \neq 1$.

1 Factoring by Common Factors

1.1 Greatest Common Factor of Integers

Recall that a **factor** of an integer is a smaller integer that divides it evenly. For example, the number 8 has factors $1,2,4$, and 8 because $1\times 8 = 2\times 4 = 8$. Given a pair of numbers, we may list out the factors of each number and see if the two lists have anything in common. For example, consider the numbers 72 and 84:

- 72 has factors $1,2,3,4,6,8,9,12,18,24,36$ and 72.
- 84 has factors $1,2,3,4,6,7,12,14,21,28,42$ and 84.

Based on these two lists, we see that 72 and 84 have some factors in common, namely: $1,2,3,4,6$ and 12. For example, 6 is a common factor of 72 and 84 because $72\div 6$ and $84\div 6$ are both whole numbers, but 7 is not a common factor of 72 and 84 because $72\div 7$ is not an integer.

When simplifying fractions, we saw the "greatest common factor", which allowed us to simplify expressions like $\frac{72}{84}$ by evenly dividing both the numerator and denominator by the largest number possible. Generally, this is quite useful: finding the **greatest** common factor (i.e. the largest common factor), allows us to make our simplification as strong as possible.

Exercise 1

Find the greatest common factor of the following pairs.

1. 10 and 50
2. 14 and 49
3. 10, 25, and 50
4. 7 and 13

1.2 Greatest Common Monomial Factor

Since we can perform multiplication and division on polynomials, we can meaningfully create the idea of "polynomial factors" as well.

Factor of a polynomial

A **factor** of a polynomial $P(x)$ is a non-zero polynomial $D(x)$ such that

$$P(x) = Q(x)\,D(x)$$

for some polynomial $Q(x)$. Equivalently: $D(x)$ is a factor of $P(x)$ if $D(x)$ divides $P(x)$ **evenly** (i.e. there is no remainder).

In this section we focus on a special (and extremely common) kind of factor: a **monomial factor**.[1] Our situation is slightly more subtle than with integers, because there is no single natural way to line up all monomials from "smallest" to "largest". So, when we say *greatest* common monomial factor, we do not mean "largest" in the familiar numerical sense. Instead, we mean the common monomial factor that contains as many powers of the variables as possible while still dividing every term. Concretely, we do the following.

Greatest common monomial factor

Given a polynomial whose terms are monomials, the **greatest common monomial factor** (GCMF) is:

- the GCF of the **integer coefficients**, and
- the highest power of each variable that appears in **every term**.

For example, consider the polynomials $8x^5$ and $16x^4$. According to our scheme defined above, the greatest common monomial factor is the GCF of the coefficients 8 and 16, which is 8, together with the highest power of x that appears in both $8x^5$ and $16x^4$, which is x^4. So, the greatest common monomial factor is $8x^4$.

For completeness, we could also verify this by simply writing out all of the monomial factors with positive integer coefficients, and then start comparing the two lists:

[1] Recall that a monomial is simply a polynomial with one term, like $6x^3$ or $-2y^2$.

$$\text{Monomial factors of } 8x^5: 1,\ 2,\ 4,\ 8,\ x,\ 2x,\ 4x,\ 8x,\ x^2,\ 2x^2,\ 4x^2,\ 8x^2,\ x^3,\ 2x^3,\ 4x^3,\ 8x^3,\ x^4,\ 2x^4,\ 4x^4,\ 8x^4,\ x^5,\ 2x^5,\ 4x^5,\ 8x^5.$$

$$\text{Monomial factors of } 16x^4: 1,\ 2,\ 4,\ 8,\ 16,\ x,\ 2x,\ 4x,\ 8x,\ 16x,\ x^2,\ 2x^2,\ 4x^2,\ 8x^2,\ 16x^2,\ x^3,\ 2x^3,\ 4x^3,\ 8x^3,\ 16x^3,\ x^4,\ 2x^4,\ 4x^4,\ 8x^4,\ 16x^4.$$

Generally, we can use the reverse of the distributive property to factor out a common monomial factor:

$$D(x)Q_1(x) + D(x)Q_2(x) = D(x)\big(Q_1(x) + Q_2(x)\big).$$

So, we may "pull out" a common factor from a sum of polynomials in order to factorization an expression.

As an example, let's consider the polynomial $6x - 18$. The greatest common factor of 6 and 18 is 6, and the highest power of x that divides both x^1 and x^0 is $x^0 = 1$. So the common monomial factor is merely 6. We can use the above "reverse distributive property" to factorization:

$$6x - 18 = 6(x) - 6(3) = 6(x - 3).$$

This can be double-checked by expanding out the factorization: $6(x - 3) = 6x - 6 \cdot 3 = 6x - 18$ which is indeed correct.

As another example, consider the polynomial $10y^3 - 25y^2$. Here, the greatest common factor of 10 and 25 is 5, and the highest power of y common to both terms is y^2. So, the greatest common monomial factor is $5y^2$. We can factorization the polynomial $10y^3 - 25y^2$ by "pulling this $5y^2$ term out":

$$10y^3 - 25y^2 = 5y^2(2y) - 5y^2(5) = 5y^2(2y - 5).$$

Again this can be double-checked by expanding out the proposed solution and comparing it to the original $10y^3 - 25y^2$.

1.3 Factoring out a negative monomial

Usually we choose the greatest common monomial factor to have a *positive* coefficient. However, sometimes it is convenient or aesthetically pleasing to factor out a negative, especially to make the leading term inside brackets positive.

For example, we could factor the polynomial $-2x^2 + 8x - 12$ in two different ways:

(a) Factorization the common monomial 2:

$$-2x^2 + 8x - 12 = 2(-x^2 + 4x - 6).$$

(b) Factorization the common monomial -2:

$$-2x^2 + 8x - 12 = -2(x^2 - 4x + 6).$$

Either solution is entirely valid, and simply boils down to a matter of taste.

Exercise 2

Factorization the greatest common monomial factor.

1. $x^2 + 3x$
2. $6x^4 + 2x^2$
3. $x^3 - x$
4. $12y^5 - 18y^3$
5. $-8x^3 + 4x^2 - 12x$
6. $15a^2b - 10ab^2$

2 Factoring Trinomials of the Form $x^2 + bx + c$

We will now consider the next least-complicated version of factorization, which is factorizing a trinomial of the form $x^2 + bx + c$. In this case, we can actually try to "solve this generally" by reverse-engineering the values of the individual factors. The key observation is that the product of two binomials often produces a trinomial:

$$(x + m)(x + n) = x^2 + (m + n)x + mn.$$

So, if we want to factor $x^2 + bx + c$, we are looking for a pair of numbers m and n that reproduce the middle and constant terms.

Factoring $x^2 + bx + c$

To factor

$$x^2 + bx + c$$

into a product of two binomials, we need to find two numbers m and n such that

$$m + n = b \qquad \text{and} \qquad mn = c.$$

Then

$$x^2 + bx + c = (x + m)(x + n).$$

Thus we have the informative rubric:

> What pair of numbers add to equal the middle term and multiply to equal the right term?

For example, consider the trinomial $x^2 + 5x + 6$. In order to factorization this, we are looking for two numbers m and n such that $m + n = 5$ and $mn = 6$. By merely listing out all the combinations, we quickly see that the desired numbers are 2 and 3. So:

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

If c is large (or negative), it helps to list factor pairs of c and test their sums by trial and error. Eventually (hopefully) one of the pairs will fit the conditions for factorization.

As another example, consider the trinomial $x^2 - 5x - 24$. Here, we are looking for a pair of numbers m and n that have opposite signs, since their product needs to equal -24. There are several pairs of factors of 24, so we may list them out and simply look for the right pair that happens to add to -5. The pairs of factors are:

$$-24 = (-1)(24),\ (-2)(12),\ (-3)(8),\ (-4)(6).$$

As it so happens, the pair that sums to -5 is 3 and -8. So, our factorization is:

$$x^2 - 5x - 24 = (x+3)(x-8).$$

Exercise 3

Factorization the following.

1. $x^2 + 7x + 12$
2. $x^2 + 2x - 15$
3. $x^2 - 9x + 20$
4. $x^2 - 3x - 18$
5. $x^2 + 11x - 26$
6. $x^2 - 13x + 36$

3 Factoring Trinomials in Two Variables

Some trinomials look like

$$x^2 + bxy + cy^2.$$

Even though there are two variables, the logic is the same as before:

$$(x+my)(x+ny) = x^2 + (m+n)xy + mny^2.$$

So we still search for two numbers m, n with

$$m + n = b, \qquad mn = c,$$

and simply make sure to remember to split up the y^2 term as the product $y \times y$ in the process.

Factoring $x^2 + bxy + cy^2$

To factor

$$x^2 + bxy + cy^2$$

we need to find numbers m, n so that

$$m + n = b, \qquad mn = c.$$

Then:

$$x^2 + bxy + cy^2 = (x+my)(x+ny).$$

As an example, consider the trinomial in two variables $x^2 - xy - 12y^2$. Here, we are looking for two numbers m and n such that $m + n = -1$ and $mn = -12$. By some process of trial and error, we see that the numbers are -4 and 3. So, the factorization is:

$$x^2 - xy - 12y^2 = (x-4y)(x+3y).$$

Exercise 4

Factorization the following.

1. $x^2 + 11xy + 10y^2$
2. $y^2 - 6xy + 8x^2$
3. $x^2 - 7xy + 12y^2$
4. $y^2 + 9xy + 20x^2$

4 Factoring Completely

Sometimes a trinomial has a common monomial factor *and* can also be factored as a trinomial afterwards. When we do all possible steps, we call this **factoring completely**.

Factoring completely

To factor a polynomial **completely**:

1. First factor out the greatest common monomial factor.
2. Then factor whatever remains (often a trinomial).
3. Repeat until nothing further can be factored using our tools.

For example, consider the complicated-looking polynomial $2x^2 - 4x - 6$. This is not a trinomial of the form $x^2 + bx + c$, so we cannot immediately use our technique of Section 2. However, we observe that we can first factor out the common factor of 2:

$$2x^2 - 4x - 6 = 2(x^2 - 2x - 3).$$

which reduces our trinomial into something that *does* have the leading coefficient equal to 1. Now, factor the remaining trinomial:

$$x^2 - 2x - 3 = (x - 3)(x + 1),$$

which gives the complete factorization:

$$2x^2 - 4x - 6 = 2(x - 3)(x + 1).$$

Exercise 5

Factorization completely.

1. $2x^2 - 12x + 10$
2. $3x^3 - 27x^2 + 54x$
3. $4y^4 - 32y^3 + 28y^2$
4. $6x^2 + 18x - 24$

5 Factoring Some Trinomials of the Form $ax^2 + bx + c$

So far we have focused on trinomials where the leading coefficient is 1, or at least situations in which we can cleverly reduce our problem to that case. Now we look at the more general, more complicated case:

$$ax^2 + bx + c \qquad \text{where } a \neq 1.$$

We are still working with a quadratic, so we can try to use "reverse FOIL", like we did in the previous section. In this case, the situation is more complicated because we need to also consider the fact that the coefficient a might not be 1, so it can also be broken into non-trivial factors. In the most general sense, we are now looking not for two numbers m and n, but for *four* numbers m, n and p, q. A general product of degree-1 polynomials will therefore look like:

$$(px + m)(qx + n).$$

We can again try to "reverse engineer" some constraints on these unknown numbers by using FOIL and comparing the answers to a, b and c:

$$(px + m)(qx + n) = pq\,x^2 + pn\,x + qm\,x + mn = pq\,x^2 + (pn + qm)\,x + mn.$$

If this polynomial is supposed to equal the general trinomial $ax^2 + bx + c$, then we need to match up all of these coefficients:

$$pq\,x^2 + (pn + qm)\,x + mn = ax^2 + bx + c,$$

which means that we need:

$$pq = a, \qquad mn = c, \qquad \text{and} \qquad pn + qm = b.$$

In practice, this is often a controlled, trial-and-error process starting with an educated guess. One way to do this is to simply list out all factor pairs of a and c, and test whether the **outer** + **inner** products ever equal b.

Factoring $ax^2 + bx + c$ (the trial-and-error method)

To factor $ax^2 + bx + c$:

1. Factorization out any common monomial factor first (if it exists).
2. List factor pairs for a (to decide the leading terms).
3. List factor pairs for c (to decide the constant terms; watch the signs).
4. Test combinations until **outer** + **inner** equals the number b.

Example

We will factor $4x^2 - 4x - 3$. Here there is no common factor, so we cannot hope to pull out something and reduce our question to something involving a trinomial with leading coefficient equal to 1. Instead, we simply accept this and we look for a product $(px + m)(qx + n)$ with $a = 4$ and $c = -3$. A good candidate structure is $(2x + \square)(2x + \square)$ because $2 \cdot 2 = 4$. Try constants whose product is -3: $(+1, -3)$ or $(-1, +3)$. Check:

$$(2x + 1)(2x - 3) = 4x^2 - 6x + 2x - 3 = 4x^2 - 4x - 3.$$

So:

$$4x^2 - 4x - 3 = (2x + 1)(2x - 3).$$

Example

We now factorization $6x^2 + 5x - 4$. Since $a = 6$, we can try something like $(2x + \square)(3x + \square)$ to start. Observe that since $c = -4$, the constants must have opposite signs. We try $(2x - 1)(3x + 4)$:

$$(2x - 1)(3x + 4) = 6x^2 + 8x - 3x - 4 = 6x^2 + 5x - 4.$$

This works, so:

$$6x^2 + 5x - 4 = (2x - 1)(3x + 4).$$

Exercise 6

Factorization the following.

1. $4x^2 + 4x - 3$
2. $6x^2 - 7x - 3$
3. $8x^2 + 2x - 3$
4. $9x^2 - 12x + 4$
5. $10x^2 + 11x - 6$
6. $12x^2 - 5x - 2$

Solutions to the Exercises

Exercise 1

1. $\text{GCF}(10, 50) = 10$.
2. $\text{GCF}(14, 49) = 7$.
3. $\text{GCF}(10, 25, 50) = 5$.
4. $\text{GCF}(7, 13) = 1$.

Exercise 2

1. $x^2 + 3x = x(x + 3)$.
2. $6x^4 + 2x^2 = 2x^2(3x^2 + 1)$.
3. $x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$.
4. $12y^5 - 18y^3 = 6y^3(2y^2 - 3)$.
5. $-8x^3 + 4x^2 - 12x = -4x(2x^2 - x + 3)$.
6. $15a^2b - 10ab^2 = 5ab(3a - 2b)$.

Exercise 3

1. $x^2 + 7x + 12 = (x + 3)(x + 4)$.
2. $x^2 + 2x - 15 = (x + 5)(x - 3)$.
3. $x^2 - 9x + 20 = (x - 5)(x - 4)$.
4. $x^2 - 3x - 18 = (x - 6)(x + 3)$.
5. $x^2 + 11x - 26 = (x + 13)(x - 2)$.
6. $x^2 - 13x + 36 = (x - 9)(x - 4)$.

Exercise 4

1. $x^2 + 11xy + 10y^2 = (x + y)(x + 10y)$.
2. $y^2 - 6xy + 8x^2 = (y - 4x)(y - 2x)$.
3. $x^2 - 7xy + 12y^2 = (x - 3y)(x - 4y)$.
4. $y^2 + 9xy + 20x^2 = (y + 4x)(y + 5x)$.

Exercise 5

1. $2x^2 - 12x + 10 = 2(x^2 - 6x + 5) = 2(x - 5)(x - 1)$.
2. $3x^3 - 27x^2 + 54x = 3x(x^2 - 9x + 18) = 3x(x - 3)(x - 6)$.
3. $4y^4 - 32y^3 + 28y^2 = 4y^2(y^2 - 8y + 7) = 4y^2(y - 1)(y - 7)$.
4. $6x^2 + 18x - 24 = 6(x^2 + 3x - 4) = 6(x + 4)(x - 1)$.

Exercise 6

1. $4x^2 + 4x - 3 = (2x + 3)(2x - 1)$.
2. $6x^2 - 7x - 3 = (2x - 3)(3x + 1)$.
3. $8x^2 + 2x - 3 = (4x + 3)(2x - 1)$.
4. $9x^2 - 12x + 4 = (3x - 2)^2$.
5. $10x^2 + 11x - 6 = (5x - 2)(2x + 3)$.
6. $12x^2 - 5x - 2 = (4x + 1)(3x - 2)$.