MAT162 Lecture 7: Sine and Cosine

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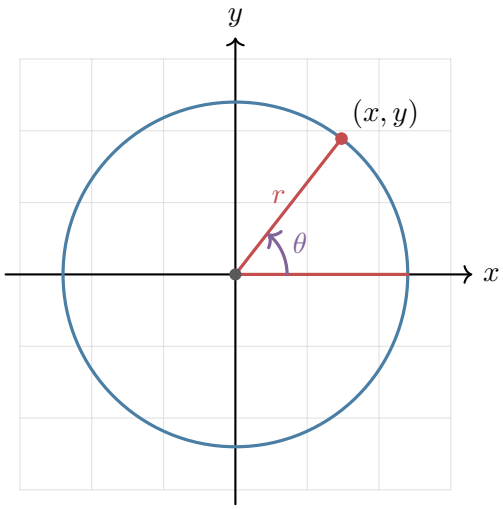
In Lecture 6 we studied the geometry of the circle. In particular, we saw that a point on a circle can be described by an angle θ, and that radians give us a natural way to measure this angle. In this lecture we will use exactly the same picture to define two important functions: sine and cosine.

We will first trace a point around the unit circle and keep track of its horizontal and vertical coordinates. We will then study several identities satisfied by sine and cosine. In the second half of the lecture we will use the same geometry to introduce polar coordinates.

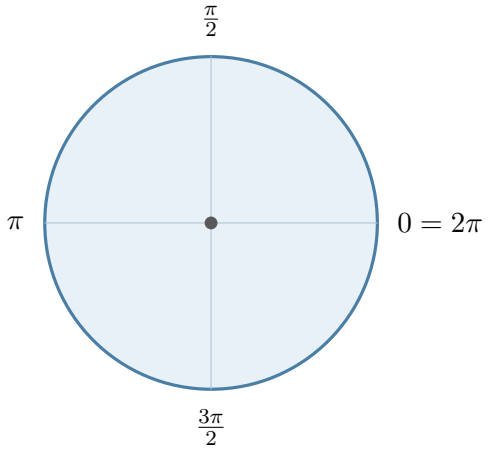
1 Sine and Cosine

1.1 A Reminder: the Circle

Recall from Lecture 6 that the circle of radius r centred at the origin is the collection of all points (x, y) that satisfy the equation $x^2 + y^2 = r^2$. As we saw last time, this equation is simply the distance formula in disguise. Therefore, by design, every point on the circle is exactly distance r away from the origin. Since the circle is a one-dimensional object, we only need a single number to describe its points. The easiest way to do this is via θ, which represents the angle between the radial line and the positive x-axis:

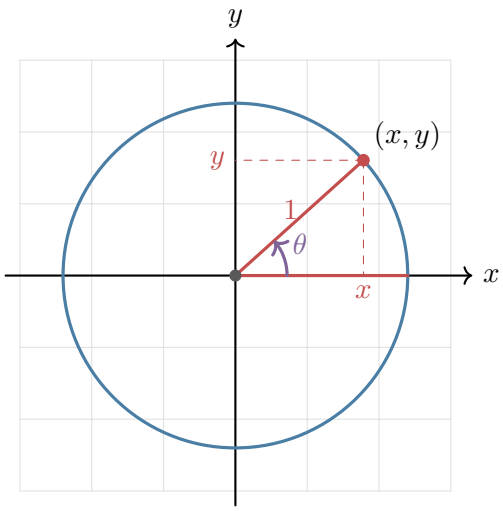


As a convention, these angles are measured anticlockwise from the positive x-axis. We will also commit ourselves to the use of radians, meaning that a full rotation around the circle adds up to 2π. Under this convention, familiar angles such as 90°, 180°, or 270° can be calculated as an appropriate ratio of 2π:



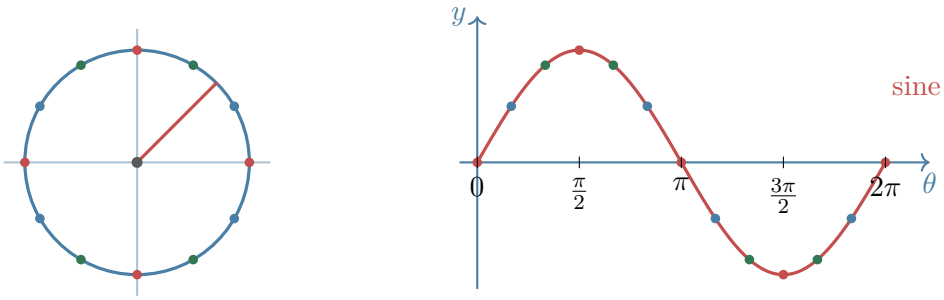
1.2 Defining Sine and Cosine

Consider now the circle of radius 1, that is, the so-called "unit circle". Since the circle sits inside $\mathbb{R}^2$, any angle θ defines a pair of Cartesian coordinates (x, y) by projecting onto the two axes:



Suppose now that we take a point, starting at $\theta = 0$, and slowly increase the value of θ so that our point rotates all the way around the circle. As we do this, we will see that the point will first climb up until it hits a peak at $\theta = \frac{\pi}{2}$, then will start to descend back down the other side of the circle. It will keep descending until it passes through the x-axis for a second time at $\theta = \pi$, and will keep going until it hits its lowest point at $\theta = \frac{3\pi}{2}$. Then, the point will start to climb back up the circle until it returns to its original starting point at $\theta = 2\pi$.

If we were to keep track of the exact height of our point as it moves around the circle, then the result would be the following graph:

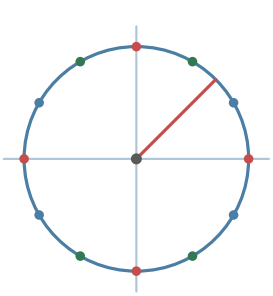


Similarly, if we were to keep track of our point's horizontal movement as it goes on its journey around the circle, then:

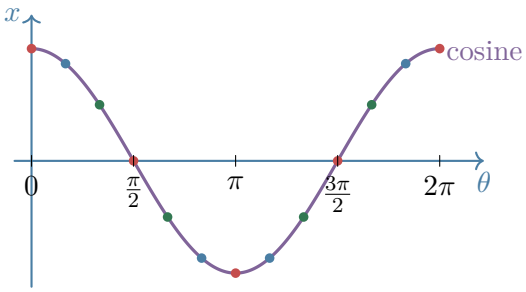
- at $\theta = 0$, our point starts with a horizontal coordinate of 1. This is because we are working with the unit circle, so its starting position is simply $(1, 0)$.
- as θ grows and the point starts to move around the circle, its horizontal displacement will become less and less. As it passes through the top of the circle at $\theta = \frac{\pi}{2}$, the point will be directly on the y-axis, and therefore will have coordinates $(0, 1)$, meaning that its horizontal displacement is exactly 0.

- Increasing θ more means that the point will start to move *away* from 0 in the x-direction. As the point moves down the other side of the circle, it will cross the x-axis at $\theta = \pi$. As it does, the point will have coordinates $(-1, 0)$.
- Continuing with its journey, the point will then move down to the bottom half of the circle. At $\theta = \frac{3\pi}{2}$ it will cross the y-axis and again will have zero horizontal displacement, i.e. $(0, -1)$.
- Finally, the point will return to its starting point of $(1, 0)$ at $\theta = 2\pi$.

Visually, we can map the horizontal displacement as a function of θ to obtain the following:



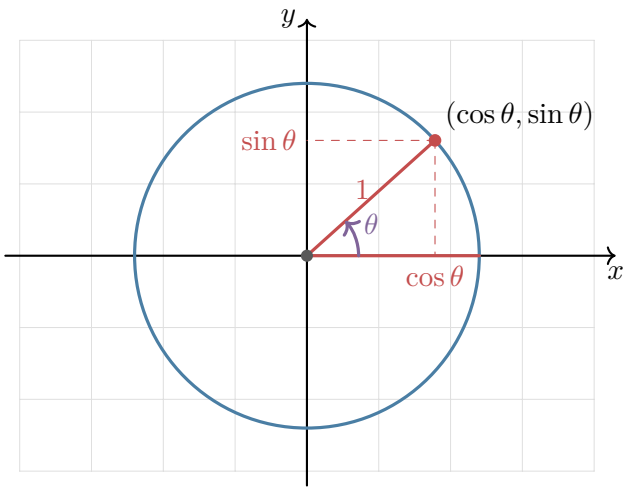
unit circle



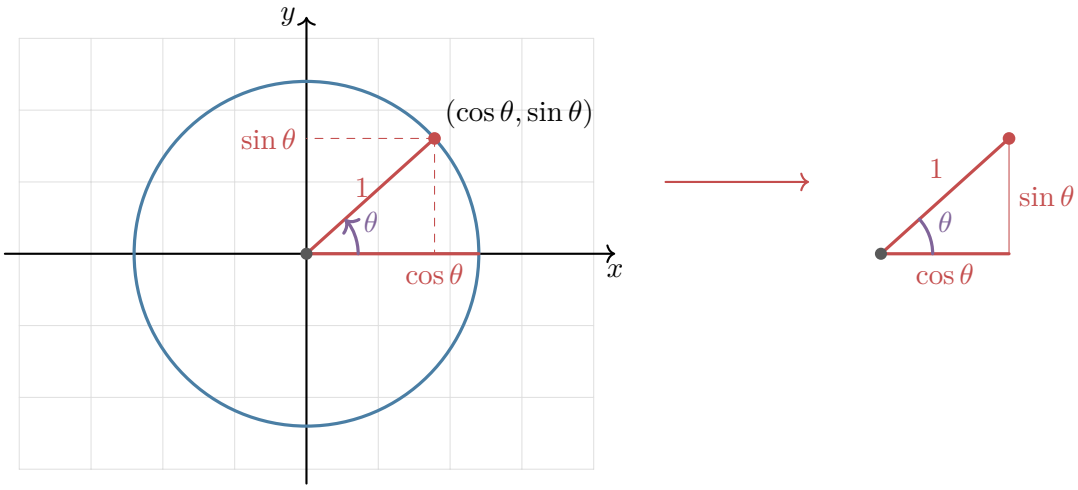
These two waves define the sine and cosine functions.

1.3 Understanding Sine and Cosine

By definition, sine and cosine are simply the vertical and horizontal coordinates of a point as it rotates around the unit circle. This means that, at any stage, the x and y components of a point are given by $\cos\theta$ and $\sin\theta$, respectively:



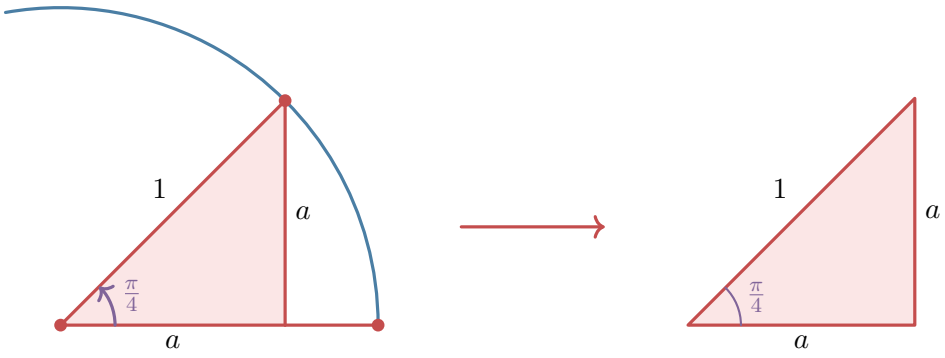
Generally speaking, we often think of sine and cosine, or trigonometry, as the study of triangles. As the diagram below illustrates, we can always create a right-angled triangle from a circle by projecting down onto the two coordinate axes:



Sine and Cosine

The point on the unit circle corresponding to the angle θ has coordinates $(\cos(\theta), \sin(\theta))$. Thus $\cos(\theta)$ is the horizontal position and $\sin(\theta)$ is the vertical position.

As a first example, suppose that we rotate a point through $\theta = \frac{\pi}{4}$, which is 45 degrees. In this case, the triangle that we create will have sides of equal length:



Pythagoras' Theorem tells us that:

$$1^2 = a^2 + a^2 \quad \Longrightarrow \quad 1 = 2a^2$$

, meaning that $a^2 = \frac{1}{2}$ and therefore, since $a > 0$, $a = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Since these two sides are precisely the sine and cosine components, we may conclude that

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Suppose that the point on the unit circle corresponding to the angle θ has coordinates (x, y). We define the horizontal coordinate to be $\cos(\theta)$ and the vertical coordinate to be $\sin(\theta)$. In other words, $(x, y) = (\cos(\theta), \sin(\theta))$.

2 Trigonometric Identities

Sine and cosine come from the same geometric picture, so the two functions are related in many different ways. Some useful identities are collected below.

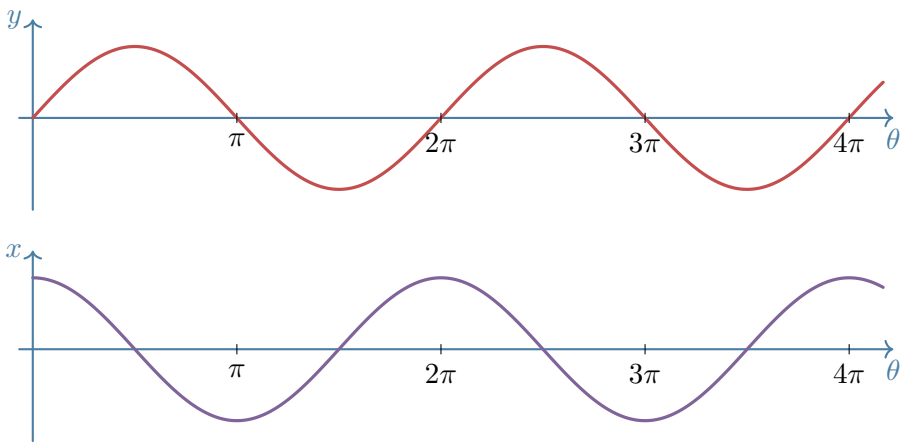
Some Trigonometric Identities

$$\sin(\theta + 2\pi) = \sin(\theta), \qquad \cos(\theta + 2\pi) = \cos(\theta),$$
$$\sin(-\theta) = -\sin(\theta), \qquad \cos(-\theta) = \cos(\theta),$$
$$\sin^2(\theta) + \cos^2(\theta) = 1,$$
$$\sin(\theta + \varphi) = \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi),$$
$$\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi),$$
$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

We will now use the geometry of the circle to unpack these identities one-by-one.

2.1 Periodicity

Suppose that we start at $\theta = 0$. When we rotate our point all the way around the circle, the point collects an angular contribution of 2π and then returns to its starting position. If we increase the angle further after this, then we will just be repeating the same orbit of the circle for a second time. This means that all of the coordinates for θ and $\theta + 2\pi$ will be the same. Graphically, this means that the sine and cosine waves will just repeat for a second time:

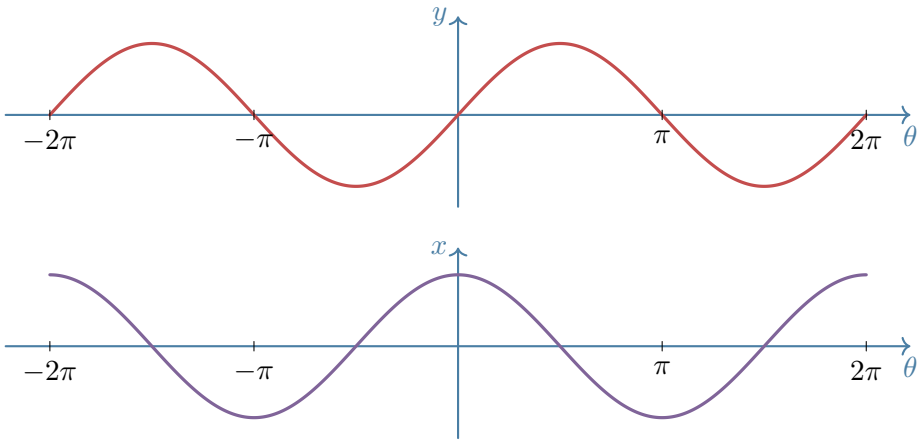


Algebraically, this explains why $\sin(\theta + 2\pi) = \sin(\theta)$ and $\cos(\theta + 2\pi) = \cos(\theta)$. Functions that repeat their values are known as *periodic*. Both sine and cosine are periodic, with period equal to 2π.

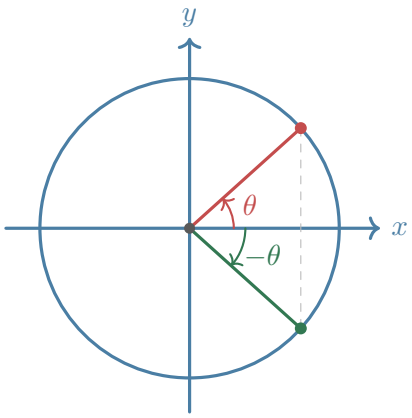
2.2 Odd and Even Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* whenever $f(x) = f(-x)$, and it is called *odd* whenever $-f(x) = f(-x)$. Graphically speaking, even functions are symmetric about the y-axis, whereas odd

functions are reflected in both the x and y axes at the same time, giving them a kind of rotational symmetry. The graphs of sine and cosine happen to have these particular symmetries:



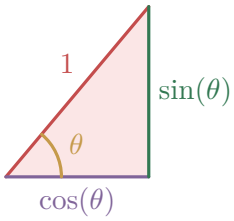
This can be seen directly from the geometry of the unit circle. To visualize negative angles, we can imagine rotating our point *backwards* around the circle. As a matter of fact, the points corresponding to the two angles θ and $-\theta$ will always have exactly the same horizontal displacement, but their vertical coordinates will have opposite signs:



Consequently, $\cos(-\theta) = \cos(\theta)$, so cosine is an even function. On the other hand, $\sin(-\theta) = -\sin(\theta)$, so sine is an odd function.

2.3 The Pythagorean Identity

At any point on the unit circle we can project vertically onto the x-axis. This creates a right triangle whose hypotenuse is 1, whose horizontal side has signed length $\cos(\theta)$, and whose vertical side has signed length $\sin(\theta)$:



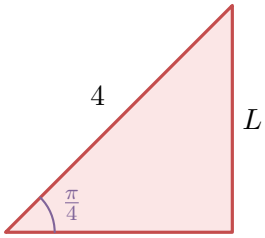
We may apply Pythagoras' Theorem to this triangle to immediately obtain:

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

The same presentation also works for a circle of radius r. In this case, the corresponding triangle has sides $r\cos(\theta)$ and $r\sin(\theta)$, so Pythagoras' Theorem tells us that

$$r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2.$$

To illustrate this further, let's consider an example. Suppose that we have a right-angled triangle as follows:

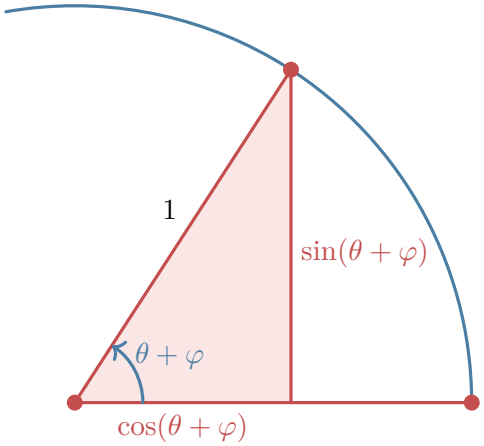


that is, the hypotenuse has length 4, and the interior angle is $\frac{\pi}{4}$. We can calculate the length of L by using $r\sin(\theta)$, so here

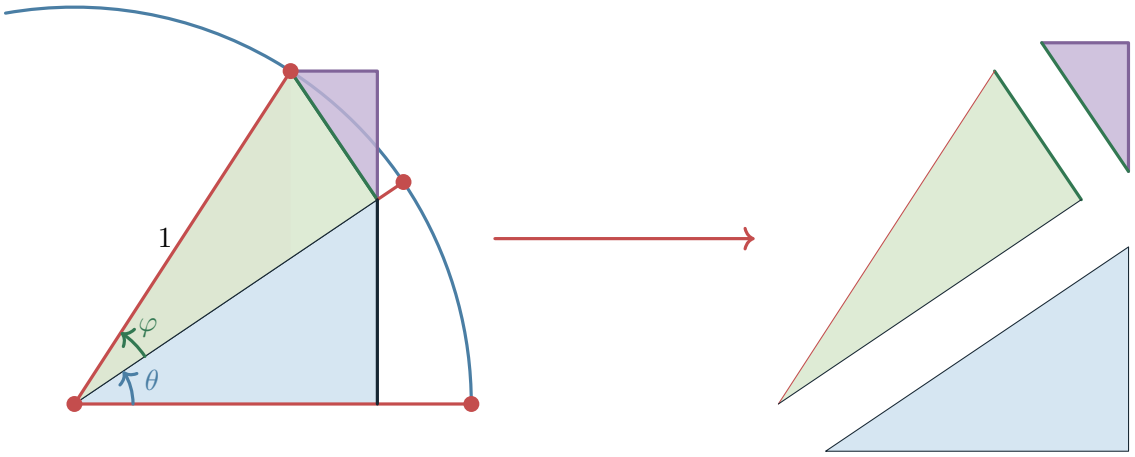
$$L = 4\sin\left(\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}.$$

2.4 Addition Formulae

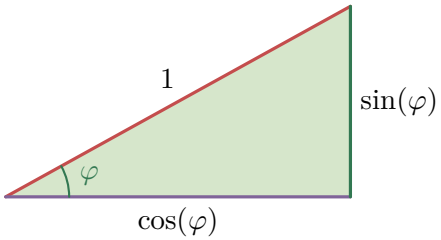
We now turn to the two identities involving $\sin(\theta + \varphi)$ and $\cos(\theta + \varphi)$. The derivation of these two identities is slightly more involved, and we will have some fun and argue them from a geometric perspective. Firstly, we note that our two expressions correspond to the lengths below:



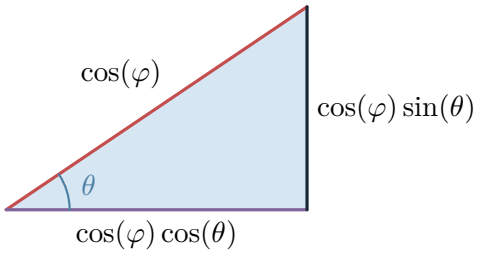
We will now figure out a clever way to describe the two lengths $\cos(\theta+\varphi)$ and $\sin(\theta+\varphi)$. Firstly, we have to note what "$\theta+\varphi$" actually means. Every point on the unit circle can be described by a rotation from the x-axis anticlockwise, according to some angle. If we have two of these angles θ and φ added together, then this will result in doing the rotation by θ *and then* doing the rotation by φ. We will now break the rotation $\theta+\varphi$ into a θ piece and a φ piece and create three important triangles:



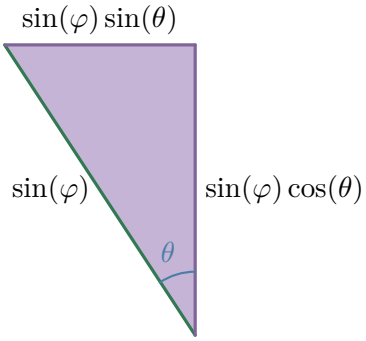
We will now describe the dimensions of each of these triangles. We start with the green triangle. Firstly, its hypotenuse is equal to 1 (since we are working with the unit circle). The internal angle is equal to φ, and therefore the two sides are given by $\cos(\varphi)$ and $\sin(\varphi)$:



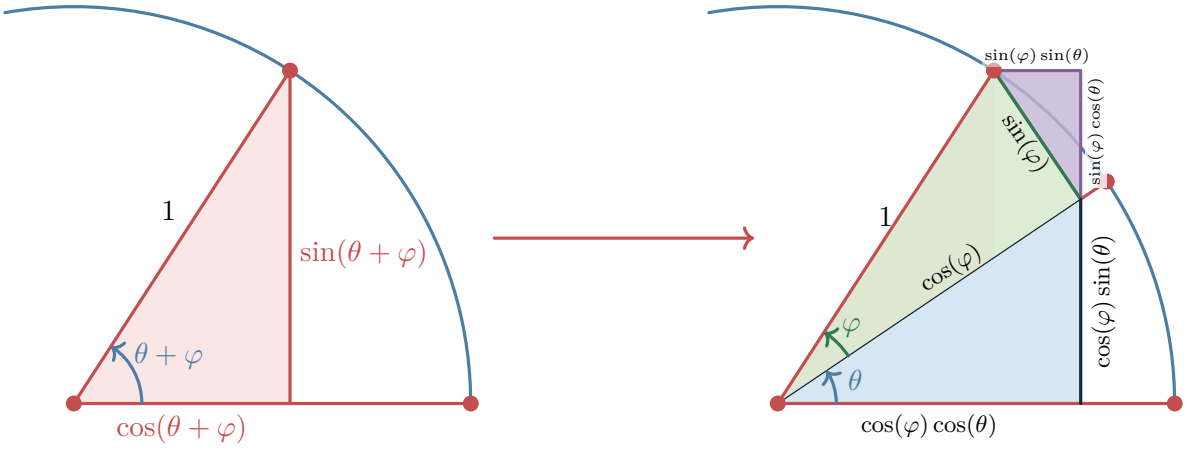
We now move on to the blue triangle. In this case, its hypotenuse is connected to the green triangle, so it has length $\cos(\varphi)$. The internal angle of the blue triangle is equal to θ, so by trigonometry, the missing side lengths of the blue triangle are $\cos(\varphi)\cos(\theta)$ and $\cos(\varphi)\sin(\theta)$:



Finally, consider the purple triangle. Its hypotenuse is connected directly to the green triangle, and therefore has length equal to $\sin(\varphi)$. Moreover, the green side lies in a right-angled triangle, meaning that after some basic calculations[1] the angle of interest inside the purple triangle is equal to θ. This gives the two remaining lengths of the purple triangle as $\sin(\varphi)\sin(\theta)$ and $\sin(\varphi)\cos(\theta)$:



We can now put the three triangles back together and compare them to our original diagram:



[1]The sum of angles of any triangle is equal to π radians, and the sum of angles on a straight line is also equal to π. The sum of the angles in the blue triangle gives us a missing piece, which also contributes equally to the angles at the vertex of the purple triangle.

By direct comparison, we see that the height of the red triangle is equal to the sum of the two smaller vertical lengths. In symbols, we have our first identity:

$$\sin(\theta + \varphi) = \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi).$$

As for the other identity, we notice that the horizontal side length of the blue triangle is the sum of $\cos(\theta + \varphi)$ and the side length of the small purple triangle. In symbols:

$$\cos(\theta)\cos(\varphi) = \cos(\theta + \varphi) + \sin(\theta)\sin(\varphi),$$

which we may rearrange to obtain the identity:

$$\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi).$$

As for the identities

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta), \qquad \text{and} \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta),$$

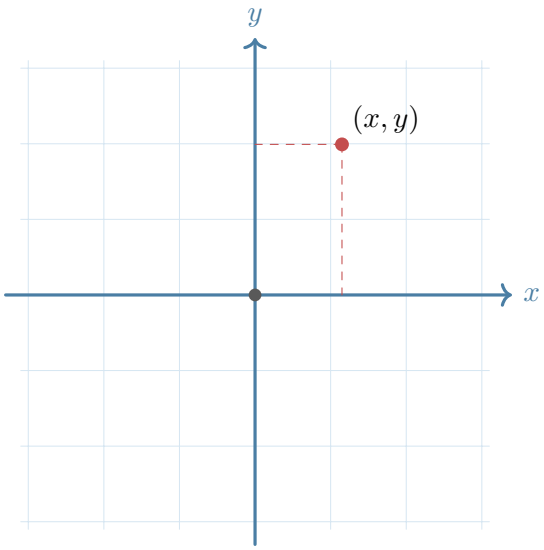
we simply set $\varphi = \theta$ and simplify.

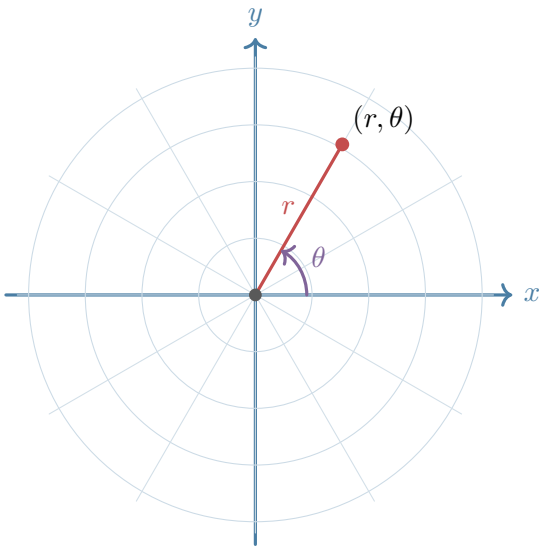
3 Polar Coordinates

3.1 Describing a Point in Two Dimensions

Consider a point in $\mathbb{R}^2$. Usually, we describe a point in relation to the origin. When working with Cartesian coordinates, these are the x and y components of the point: we impose a horizontal and a vertical direction, and then the pair (x, y) gives us instructions for how to move from the origin over to the point.

However, in principle, these are not the *only* instructions that we could use in order to describe a given point. In Lecture 5 we saw that we can perform coordinate transformations (which were represented by 2×2 matrices) and these would create other grids in $\mathbb{R}^2$. We will now describe something totally different: polar coordinates. Consider the diagram below.





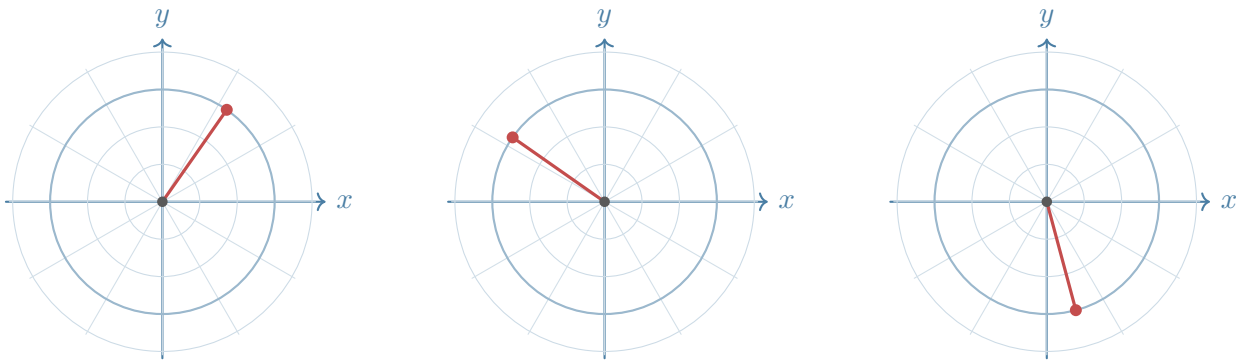
In this diagram, we have two copies of $\mathbb{R}^2$, with the same point drawn twice. On the left-hand side, we have the Cartesian description of the point: $p = (x, y)$. On the right-hand side, we have described the point in terms of a *radius* and an *angle*: $p = (r, \theta)$. We can imagine the latter as a different style of instruction: to find p within $\mathbb{R}^2$, we move a distance r away from the origin along the positive x-axis, and then we rotate around until we land on p.

Polar Coordinates

A point in the plane can be described by the pair (r, θ), where r is its distance from the origin and θ is its angle measured from the positive x-axis.

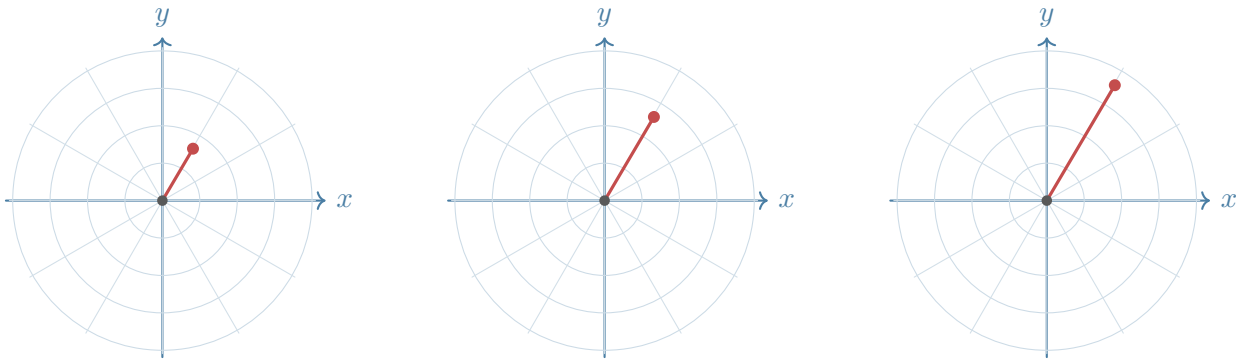
In fact, we can do this for any point in $\mathbb{R}^2$, because every point p has a distance from the origin, and therefore lies on a circle of radius r. In more poetic terms, we can think of $\mathbb{R}^2$ as the disk of infinite radius.

In the picture above, the circles and rays form a type of non-linear grid: they are rays of constant θ or circles of constant r. If we keep r fixed and change θ, then the point moves around a circle:



same r, different θ

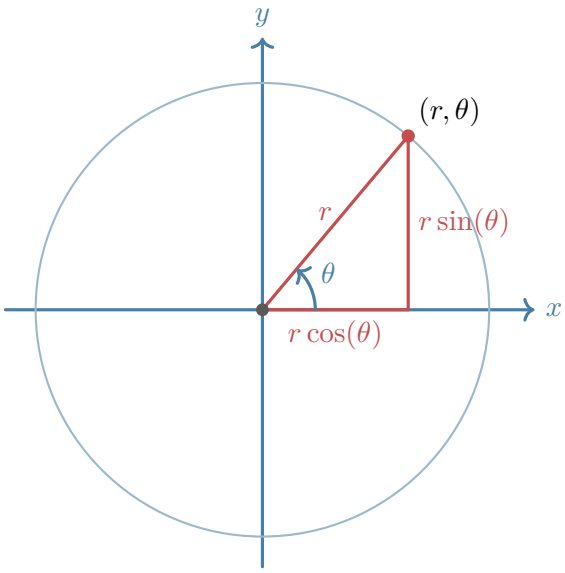
Alternatively, if we keep θ fixed and change r, the point moves towards or away from the origin along a ray:



same θ, different r

3.2 Converting to Cartesian Coordinates

Suppose that a point p has polar coordinates equal to (r, θ). Since p lies on a circle in two-dimensional space, we may project the point onto the x-axis to create exactly the same right triangle that we used earlier, except that the hypotenuse now has length r rather than 1:



Therefore, according to Section 2.3, the point p can be described in Cartesian coordinates as $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The distance formula also gives $r^2 = x^2 + y^2$.

Polar and Cartesian Coordinates

For a point $p = (r, \theta)$ written in polar coordinates, we can convert p into Cartesian coordinates (x, y) using:

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

To demonstrate how useful this is, consider the following intentionally unattractive equation:

$$x^4 + 2x^2y^2 + y^4 = 16.$$

We may factorize the left-hand side of this equation to yield $(x^2+y^2)^2 = 16$. Since $x^2+y^2 = r^2$, this becomes $r^4 = 16$. Because r is a distance and is therefore non-negative, we obtain the far simpler equation $r = 2$.