MAT162 Lecture 9: Exponentials

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In the first eight lectures we studied the real number line, straight lines, vectors, circles and trigonometric functions. We will now move on to another important collection of functions: the so-called *exponential functions*. We begin by reviewing the algebra of exponents, and extending the familiar rules from natural numbers to integers, and then to rational and even to irrational numbers. We will then investigate the graphs of exponential functions and see how they describe growth and decay, which can be used to model processes like compound interest. We will finish the lecture by using the compound interest formula to introduce a rather special number, known as e. As we will see in Lecture 10, this number has some surprising properties that give us some clues about the calculus to come.

1 The Algebra of Exponents

1.1 Multiplication and Exponentiation

Recall that multiplication by a natural number is simply a shorthand for repeated addition. For example, if we take a number a and add it to itself over and over again, say n times, then the answer is:

$$na = \underbrace{a + a + \cdots + a}_{n \text{ times}}.$$

Similarly, we may multiply a number a by itself over and over again to obtain:

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}.$$

Here, the number a is typically called the *base*, and the number n is called the *exponent*. In the same sense that multiplication is repeated addition, we may think of exponentiation as repeated multiplication.

We will now introduce several rules that allow us to manipulate algebraic expressions containing exponents. The following is a summary of the basic rules.

Rules of Exponents

For positive integers m, n and non-zero real numbers a, b, we have

$$a^m a^n = a^{m+n}, \qquad \frac{a^m}{a^n} = a^{m-n},$$
$$(a^m)^n = a^{mn}, \qquad (ab)^n = a^n b^n.$$

It should be noted that all of these rules can be obtained directly from the definition of exponents, once zero and negative exponents are defined. For instance, we may derive the first rule $a^m a^n = a^{m+n}$ by writing out the definition of a^m and a^n and then multiplying them together:

$$a^m a^n = \underbrace{(a \cdots a)}_{m \text{ times}} \underbrace{(a \cdots a)}_{n \text{ times}} = \underbrace{a \cdots a \cdot a \cdots a}_{(m+n) \text{ times}} = a^{m+n}.$$

The other rules can be similarly derived – it is a good exercise to try and see how.

1.2 Zero and Negative Exponents

At first glance, the expression a^0 is a little confusing: what could it mean to multiply a number by itself *zero* times in a row? In order to figure this out, we may use the rule $a^{m-n} = \frac{a^m}{a^n}$. If we set $m = n$, then we get:

$$a^0 = a^{m-m} = \frac{a^m}{a^m}$$

The denominator and numerator of this fraction are both the same, so they cancel each other out, giving $a^0 = 1$.

Using similar logic, we can also try to interpret negative exponents. Suppose, for instance, that we consider a^{-1}. We want this number to be the multiplicative inverse of a, so it should multiply with

a to give 1. Using this, we see that:

$$1 = a \cdot a^{-1} \quad \Longrightarrow \quad a^{-1} = \frac{1}{a}$$

More generally, we can also derive a rule for larger negative exponents:

$$1 = a \cdot a^{-1} \quad \Longrightarrow \quad 1^m = (a \cdot a^{-1})^m \quad \Longrightarrow \quad 1 = a^m \cdot (a^{-1})^m \quad \Longrightarrow \quad \frac{1}{a^m} = a^{-m}$$

Zero and Negative Exponents

For any non-zero real number a and any positive integer m,

$$a^0 = 1, \qquad a^{-1} = \frac{1}{a}, \qquad a^{-m} = \frac{1}{a^m}.$$

These rules can be used to simplify expressions that appear to be quite complicated. For example, consider the expression

$$\frac{2^5 \cdot 2^{-3}}{2^4}.$$

We can simplify this by applying exponent rules step-by-step. There may be many ways to do this; one example would be:

$$\frac{2^5 \cdot 2^{-3}}{2^4} = \frac{2^5}{2^4 \cdot 2^3} = \frac{2^5}{2^7} = 2^{-2} = \frac{1}{4}$$

Use $a^{-m} = \frac{1}{a^m}$ — Use $a^m a^n = a^{m+n}$ — Use $\frac{a^m}{a^n} = a^{m-n}$

Other examples of simplifications are:

$$\frac{(3^2)^4}{3^5} = \frac{3^8}{3^5} = 3^{8-5} = 3^3 = 27,$$

$$\left(\frac{2a^3}{b^2}\right)^2 = \frac{2^2 a^6}{b^4} = \frac{4a^6}{b^4}.$$

Note in the final expression we have assumed that $b \neq 0$, since otherwise we would be accidentally dividing by zero.

1.3 Exponents of Variables

In Lecture 1 we briefly discussed variables such as x or y, which served as placeholders for numbers. Since our rules of exponents work for the numbers considered so far, we may use exactly the same laws for variables as well, provided the expressions are legally defined. For instance, we may manipulate something like:

$$\frac{x^5 y^{-2}}{x^{-3} y^4}$$

by applying the same rules as before:

$$\frac{x^5y^{-2}}{x^{-3}y^4} = x^{5-(-3)}y^{-2-4} = x^8y^{-6} = \frac{x^8}{y^6},$$

Use $\frac{a^m}{a^n} = a^{m-n}$ — Perform the sums — Use $a^{-m} = \frac{1}{a^m}$

provided that we assume $x, y \neq 0$. This extra assumption is entirely necessary, since if either x or y were zero, then the original expression would involve a division by zero and therefore would not be defined.

1.4 Rational Exponents

We may also extend the rules of exponents to include rational numbers. As an example, assuming $x, y > 0$, we may manipulate:

$$\frac{x^{\frac{3}{2}}y^{-\frac{1}{3}}}{x^{-\frac{1}{2}}y^{\frac{2}{3}}} = x^{\frac{3}{2}-(-\frac{1}{2})}y^{-\frac{1}{3}-\frac{2}{3}} = x^2y^{-1} = \frac{x^2}{y}.$$

Similarly,

$$\frac{(2x^{\frac{1}{2}}y^{\frac{2}{3}})^3}{4x^{-\frac{1}{2}}y} = \frac{8x^{\frac{3}{2}}y^2}{4x^{-\frac{1}{2}}y} = 2x^2y.$$

According to the rules of exponents, for any non-negative number a we ought to have:

$$(a^{1/2})^2 = a^{(1/2)\cdot 2} = a,$$

meaning that $a^{1/2}$ should be a number whose square equals a. In other words, $a^{\frac{1}{2}}$ must be equal to the *square root* $\sqrt{a}$. If $a \geq 0$, we define $a^{1/2}$ to be the non-negative square root of a:

$$a^{1/2} = \sqrt{a}.$$

In exactly the same spirit, $a^{1/3}$ is a number whose cube equals a, so that $a^{1/3} = \sqrt[3]{a}$. We can generalize this as follows.

Rational Exponents

For $a > 0$, positive integers n, and integers m, we define

$$a^{1/n} = \sqrt[n]{a}, \qquad a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

Under this reading, we can interpret radicals such as square roots or cube roots as exponents that are subject to the same rules we discussed previously. Consequently, this allows us to easily manipulate expressions involving radicals. As a first example, consider the expression $\sqrt{12x^2y^3}$. Assuming that $x, y > 0$, we can break down the radical into a part which is a perfect square and separate it from the rest:

$$\sqrt{12x^2y^3} = \sqrt{4x^2y^2 \cdot 3y} = \sqrt{4x^2y^2}\sqrt{3y} = 2xy\sqrt{3y}.$$

Decompose into a piece that will cancel with the square root — Use $\sqrt{ab} = \sqrt{a}\sqrt{b}$ — Use $\sqrt{a^2} = a$

1.5 Irrational Exponents

We will now extend our ideas one step further by considering exponents of *irrational* numbers. To illustrate the idea, we will answer the question:

$$\text{What is } 2^{\pi}?$$

At first glance, this appears to be ridiculous. We know that 2^3 is a way of writing down the value of 2 multiplied by itself 3 times, and 2^4 represents 2 multiplied by itself 4 times. So, how can we multiply 2 by itself π many times? Without getting too philosophical, we will simply have to admit that this is not the correct interpretation. Instead, we will construct 2^{π} in a totally different way.

Recall first that π is an irrational number, with a never-ending decimal expansion $\pi = 3.1415....$ Each digit in this expression is a rational number:

$$\begin{aligned}\pi &= 3.1415\ldots \\ &= 3 + 0.1 + 0.04 + 0.001 + 0.0005 + \cdots \\ &= 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \cdots .\end{aligned}$$

Since each of these decimal pieces is rational, we already know what it means to raise 2 to those powers. Therefore, we can approach our desired number 2^{π} by building better and better approximations:

$$\begin{aligned}2^3 &= 8, \\ 2^{3.1} &= 2^3 \cdot 2^{\frac{1}{10}} \approx 8.57419, \\ 2^{3.14} &= 2^3 \cdot 2^{\frac{1}{10}} \cdot 2^{\frac{4}{100}} \approx 8.81524, \\ 2^{3.141} &= 2^3 \cdot 2^{\frac{1}{10}} \cdot 2^{\frac{4}{100}} \cdot 2^{\frac{1}{1000}} \approx 8.82135.\end{aligned}$$

Roughly speaking, if we were to keep going through this process, then we would keep multiplying by 2 to the power of smaller and smaller numbers. As we go through the further digits of π, these small numbers will get closer and closer to 0, meaning that 2^a will get closer and closer to 1. Therefore, when we include these into our approximation, they will influence the result less and less. In this case, our approximation will settle down towards a specific number. This number happens to be about:

$$2^{\pi} \approx 8.8249778$$

In fact, this value makes quite a lot of sense: since $3 < \pi < 4$, the value of 2^{π} lies between $2^3 = 8$ and $2^4 = 16$.

This approach of "sneaking up" on an irrational exponent via rational ones is known as a *limiting* argument. We will investigate this idea in great detail in Lecture 13.

1.6 Graphs of Exponential Functions

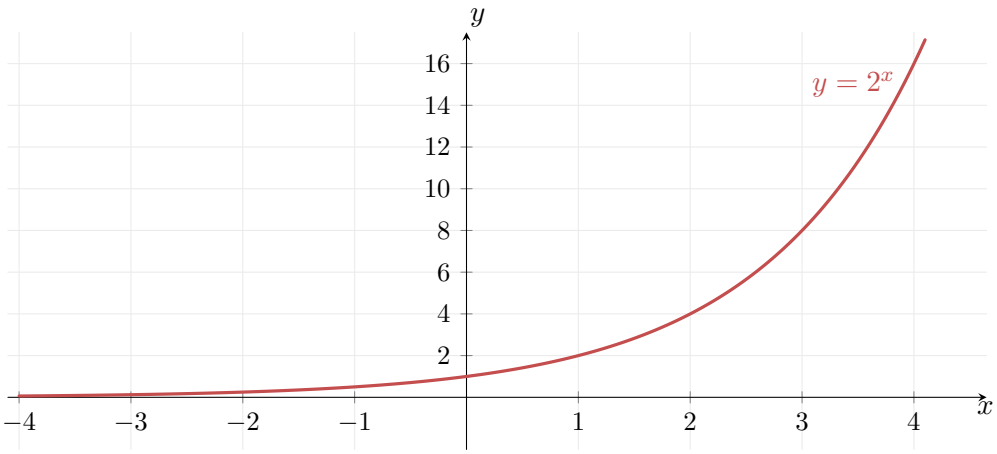
Throughout the past few sections we have demonstrated that exponents may be defined for natural numbers, integers, rational numbers and even irrational numbers. As such, if we are able to understand a quantity 2^a for any number a in $\mathbb{R}$, then we may meaningfully write a genuine function:

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{where} \qquad f(x) = 2^x.$$

This function takes any real number as an input, and then raises 2 to the power of that number. For instance,

$$f(3) = 2^3 = 8, \qquad f\left(\frac{1}{4}\right) = 2^{\frac{1}{4}} = \sqrt[4]{2} \qquad \text{and} \qquad f(-2) = 2^{-2} = \frac{1}{4}.$$

The graph of this function is depicted below.



As you can see, the graph displays two major regimes: firstly, on the left hand side the graph gets closer and closer to zero. This is because more and more negative exponents create smaller and smaller numbers:

$$\text{if } m < n \text{ then } 2^{-n} = \frac{1}{2^n} < \frac{1}{2^m} = 2^{-m}.$$

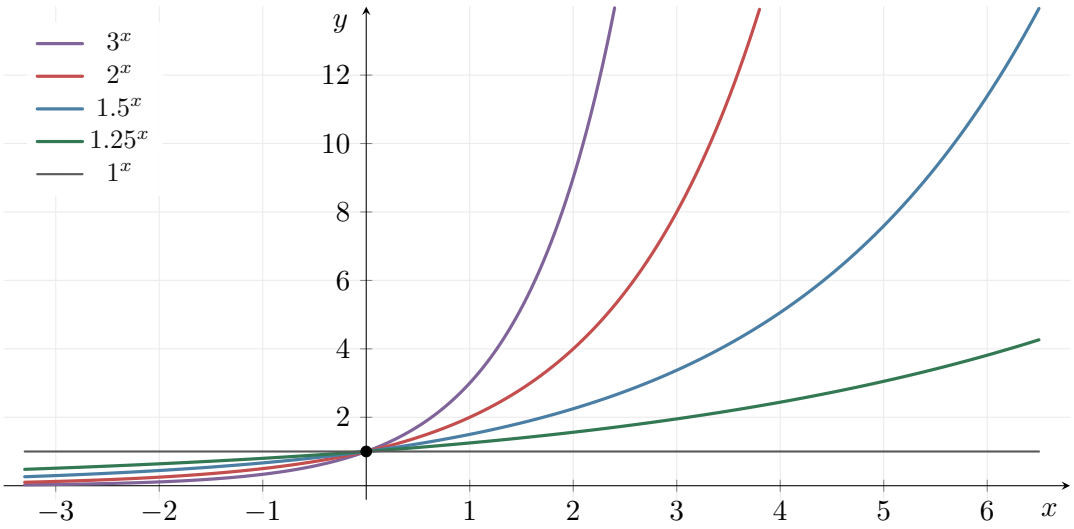
However, the graph will never actually reach zero – that would imply $2^{-n} = 0$, which is impossible.[1] On the other hand, the right hand side of the graph will get arbitrarily large as we increase the size of the exponent, since each discrete step from 2^n to 2^{n+1} doubles the height.

In general, if $b > 0$ we can define an *exponential function* $y = b^x$. The graph below depicts various examples for real x:

[1] Specifically, if it were the case that $2^{-n} = 0$, then $\frac{1}{2^n} = 0$, so multiplying both sides by 2^n would give:

$$2^n \cdot \frac{1}{2^n} = 2^n \cdot 0 \quad \Rightarrow \quad 1 = 0,$$

which is impossible.



All of these graphs cross the y axis at the value $y = 1$, since $b^0 = 1$ for every positive base b.

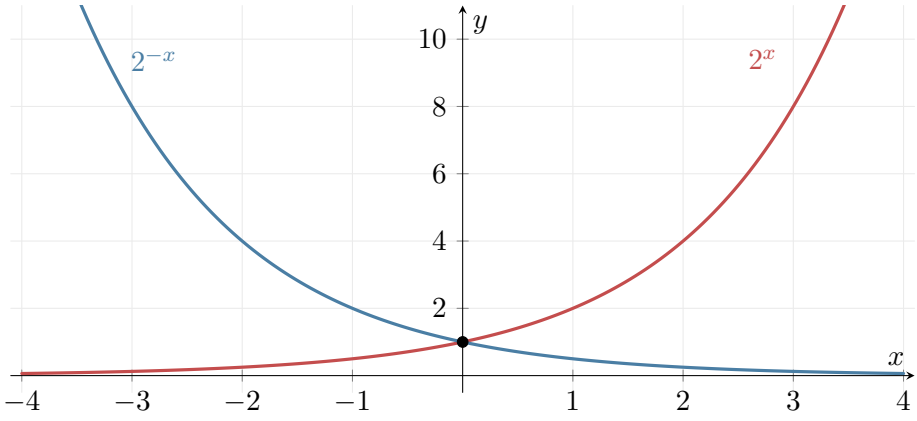
Consider now a base b less than 1, say $b = \frac{1}{2}$. In this case, our exponential function would take the form:

$$f(x) = \left(\frac{1}{2}\right)^x,$$

meaning that every time x increases by 1, the output gets divided by 2. As such, the output shrinks as x grows. Given the exponent rule

$$\left(\frac{1}{2}\right)^x = 2^{-x},$$

we can see that the graph of $y = \left(\frac{1}{2}\right)^x$ will simply reflect the graph of $y = 2^x$ across the y-axis:



Exponential Growth and Decay

For a positive base b, the function $y = b^x$ satisfies the following:

- if $b > 1$, then $y = b^x$ represents exponential growth;
- if $0 < b < 1$, then $y = b^x$ represents exponential decay;
- if $b = 1$, then $y = 1$ is a constant function.

1.7 Further Examples

We will now return to a few more involved examples of exponents. Firstly, we consider the prospect of manipulating expressions containing multiple exponents that have different bases. For instance, suppose that we have the expression:

$$2^{2x}4^x.$$

To simplify this, we may first notice that 4 happens to be equal to 2^2. Therefore we can first replace 4 with 2^2, and from there we unlock the exponent rules that can be used to simplify everything down:

$$2^{2x}4^x = 2^{2x}(2^2)^x = 2^{2x}2^{2x} = 2^{4x} = (2^4)^x = 16^x.$$

Similarly, the bases 3 and 9 are related because $9 = 3^2$. This gives

$$3^{x^2}9^x = 3^{x^2}(3^2)^x = 3^{x^2+2x} = 3^{x(x+2)}.$$

Again, the strategy here is to try and rewrite everything in terms of a common base, *and then* use the familiar exponent rules to simplify.

As our next example, let's consider something like:

$$3^{\sin^2(\theta)} \cdot 3^{\cos^2(\theta)}.$$

In order to simplify this, we can make use of the Pythagorean identity of Lecture 7:

$$3^{\sin^2(\theta)} \cdot 3^{\cos^2(\theta)} = 3^{\sin^2(\theta)+\cos^2(\theta)} = 3^1 = 3.$$

As another example, this time we make use of the identity $1 + \tan^2(\theta) = \sec^2(\theta)$ to simplify:[2]

$$\begin{aligned}\frac{3^{1+\tan^2(\theta)}}{3^{\sec^2(\theta)-1}} &= 3^{1+\tan^2(\theta)-(\sec^2(\theta)-1)} \\ &= 3^{1+\tan^2(\theta)-\tan^2(\theta)} = 3.\end{aligned}$$

In principle, we may combine this idea with the previous technique to simplify complicated-looking expressions such as:

$$\begin{aligned}2^{2\sin^2(\theta)} \cdot 4^{\cos^2(\theta)} &= 2^{2\sin^2(\theta)} \cdot 2^{2\cos^2(\theta)} \\ &= 2^{2(\sin^2(\theta)+\cos^2(\theta))} = 2^2 = 4.\end{aligned}$$

2 Defining the Number e

So far we have been exploring the general idea of exponents. We will now introduce a very specific number that finds application all over mathematics and beyond. To find this special number, we will first take a detour through some basics of finance.

We will now introduce a number which turns up naturally whenever we study exponential growth. Curiously, this number will also become one of the central objects of calculus. To find it, we will consider a very familiar application of growth: interest paid on money.

[2] Note that the final trigonometric expression is understood only where $\tan(\theta)$ and $\sec(\theta)$ are defined, i.e. the entire algebra only works when θ is not an odd multiple of $\frac{\pi}{2}$.

2.1 Simple and Compound Interest

Suppose that we deposit an initial amount P in a bank, and that the bank pays interest at a yearly rate r. In the simplest possible model, the interest is always calculated using the *original* amount of money. For example, if we were to invest 1000 dollars at an interest rate of 10% per year, then the bank would pay 100 dollars every year. This means that our money grows by the same amount each year. Consequently, after t years, our total amount is

$$A = 1000 + 100t = 1000(1 + 0.1t).$$

More generally, this gives the familiar formula for simple interest.

Simple Interest

If P is the initial investment, r is the interest rate written as a decimal, and t is the time measured in years, then the amount of money A that is accumulated according to simple interest can be calculated using the formula:

$$A = P(1 + rt).$$

For instance, an initial investment of 1000 dollars at 10% simple interest will grow after five years to

$$A = 1000(1 + 0.1 \cdot 5) = 1000(1.5) = 1500.$$

As a matter of fact, many interest calculations in the real world don't follow the formula of simple interest. In reality, it is more common for the bank to pay interest back into the same account, so that this extra earning is included in the principal investment for the next year. The year after that, the interest generated would take into account the earnings of the earnings, and so on. This idea is known as *compound interest*, because the interest is calculated by combining the principal investment and the interest already earned.

We can derive the formula for compound interest by repeatedly applying the formula for simple interest. To this end, suppose that our initial investment is P_0 and our yearly interest rate is r. After one year, the new amount is

$$P_1 = P_0(1 + r).$$

To find the compound interest earned after another year, we apply the simple-interest formula for a second time, this time starting with the new balance P_1 instead of P_0. After some simplification, we obtain:

$$P_2 = P_1(1 + r) = P_0(1 + r)(1 + r) = P_0(1 + r)^2.$$

If we continue this process for another year, we will obtain:

$$P_3 = P_2(1 + r) = P_0(1 + r)^3,$$

and the same argument works for any number of years. In general, this process will produce a formula that is *exponential* in time, instead of linear. The result is the following formula for compound interest.

Compound Interest

If an initial investment P earns compound interest at a yearly rate r, compounded once per year, then the amount after t whole years is

$$A = P(1+r)^t.$$

For whole-number values of t, this comes directly from repeated multiplication by $1+r$.

To illustrate, suppose that we deposit 1000 dollars at a yearly interest rate of 10%. After two years, we have

$$A = 1000(1.1)^2 = 1000 \cdot 1.21 = 1210.$$

After 5 years of compound interest, our new balance will be

$$1000(1.1)^5 \approx 1611,$$

which is larger than the 1500 obtained from simple interest.

2.2 Compounding More Frequently

So far we have been assuming that new interest is accumulated once per year. Suppose now that the bank decides to pay interest twice per year instead of once. How should we calculate the total interest after one year? Firstly, we note that if r is the annual interest rate, then a payment after half a year will generate half of this interest, meaning that after 6 months our principal investment will be multiplied by the quantity:

$$1 + \frac{r}{2}.$$

After a year we will have received two of these compound payments, giving a multiplier of:

$$\left(1 + \frac{r}{2}\right)^2.$$

If we keep repeating this process for t years, our general formula for the accumulated amount will become

$$A(t) = P\left(1 + \frac{r}{2}\right)^{2t}.$$

We can repeat this argument for further and further subdivisions of the year. The following formula describes the total balance we would have if interest is compounded n times per year over t years.

Compound Interest at Different Frequencies

Let P be the principal investment, r the annual interest rate, and t the time in years. If interest is compounded n times per year, then the accumulated money after a whole number of compounding periods can be calculated using the formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

2.3 The Exponential Number

For the sake of simplicity, let's look at the previous formula in the case that we invest 1 dollar at an annual interest rate of 100%, and we wait one year. In other words, we set $P = 1, r = 1$ and $t = 1$. If the interest is generated once per year, then our accumulation A will be equal to simply 2 dollars – the dollar we started with, as well as the extra dollar we earned at a 100% interest rate.

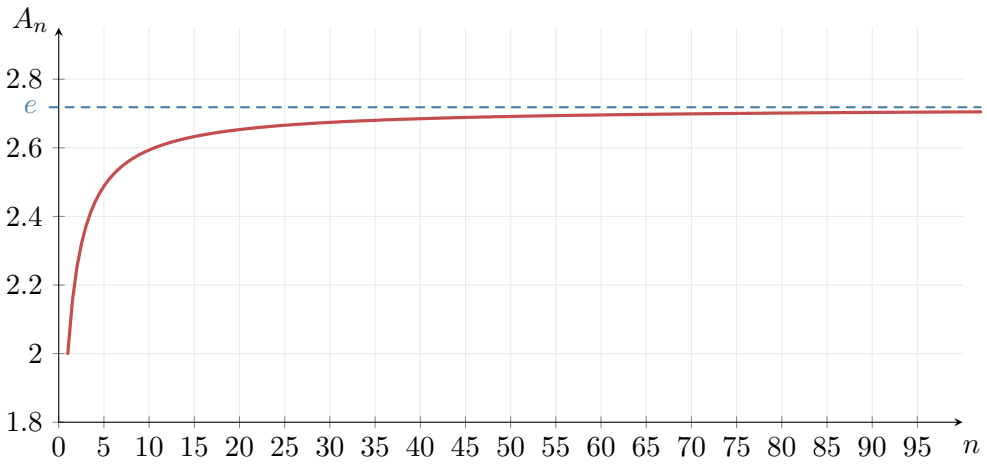
What happens if the bank compounds interest more and more frequently? In this conveniently simple setting, our formula reduces to:

$$A_n = \left(1 + \frac{1}{n}\right)^n.$$

We will now substitute different values of n and see what happens:

Frequency	**Number of payments** n	**Amount** A_n
Once per year	1	2
Every six months	2	2.25
Every month	12	2.61304
Every week	52	2.69260
Every day	365	2.71457
Every hour	8760	2.718126692
Every minute	525600	2.718279243
⋮	⋮	⋮

It appears as though increasing n doesn't lead to infinite money, and instead the answers start to converge to something. As a matter of fact, we have arrived at our special number: e. We can picture it as follows:



The red curve gets closer and closer to the dashed blue line as n grows. The height of this line is the value of e, which is about 2.71828 . . ..

The Number e

The exponential number e is defined as the value that the expression

$$\left(1+\frac{1}{n}\right)^n$$

converges to as we make n larger and larger.

Just as π naturally appeared when we studied the geometry of circles, the number e appears when we study growth by increasingly frequent compounding effects. Both are irrational numbers, and both will be related to each other in Lecture 12.

2.4 Algebra with e

We have introduced e through a rather long-winded discussion of compound interest. But we should keep in mind that e is simply a positive real number, equal to about 2.7. As such, e satisfies exactly the same exponent rules as every other positive real number:

$$e^0 = 1, \qquad e^x e^y = e^{x+y}, \qquad (e^x)^y = e^{xy}.$$

As an example, let us simplify the following expression:

$$\frac{e^{3x}\cdot e^{2-x}}{e^{x+1}} = \frac{e^{2x+2}}{e^{x+1}} = e^{2x+2-(x+1)} = e^{x+1}.$$

As another example, consider:

$$\frac{(e^{\frac{x}{2}})^4}{e^{-x}} = \frac{e^{2x}}{e^{-x}} = e^{2x-(-x)} = e^{3x}.$$

Finally, recall from Lecture 7 that $\cos(2\theta) = 1 - 2\sin^2(\theta)$. This gives another example involving our earlier trigonometry:

$$\begin{aligned} e^{2\sin^2(\theta)}\cdot e^{\cos(2\theta)} &= e^{2\sin^2(\theta)+\cos(2\theta)} \\ &= e^{2\sin^2(\theta)+1-2\sin^2(\theta)} = e. \end{aligned}$$